Private Communication in Competing Mechanism Games*

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May 28, 2019

Abstract

We study games in which principals simultaneously post mechanisms in the presence of several agents. We evaluate the role of principals' communication in these settings. As in Myerson (1982), each principal may generate incomplete information among agents by sending them private signals. We show that this channel of communication, which has not been considered in standard approaches to competing mechanisms, has relevant strategic effects. Specifically, we construct an example of a complete information game in which (multiple) equilibria are sustained as in Yamashita (2010) and none of them survives in games in which all principals can send private signals to agents. The corresponding sets of equilibrium allocations are therefore disjoint. The role of private communication we document may hence call for extending the construction of Epstein and Peters (1999) to incorporate this additional element.

Keywords: Competing Mechanisms, Private Communication.

JEL Classification: D82.

*We would like to thank Dino Gerardi, Thomas Mariotti, Mike Peters, Cristián Troncoso Valverde and Takuro Yamashita for extremely valuable feedbacks. The comments of the Editor Alessandro Pavan and of three anonymous reviewers have been extremely helpful to improve the paper. We also thank seminar audiences at European University Institute, Università degli Studi di Roma Tor Vergata, Vancouver School of Economics, Université du Luxembourg as well as conference participants at 2017 International Conference on Game Theory at Stony Brook and at 2018 North-American meeting of the Econometric Society at UC Davis for many useful discussions. Financial support from the SCOR-TSE Chair and from MIUR (PRIN 2015) is gratefully acknowledged.

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1 Introduction

We study competing mechanism games: principals compete through mechanisms in the presence of several agents. Such a strategic scenario has become a reference framework to model competition in a large number of market settings.[1]

As first pointed out by McAfee (1993) and Peck (1997), the equilibrium allocations derived in these contexts crucially depend on the set of mechanisms that principals are allowed to post. Typically, letting agents communicate to principals additional information on top of their exogenous types supports additional allocations at equilibrium.[2] This raises the issue of identifying a class of mechanisms inducing agents to reveal all their available information. In an important contribution, Epstein and Peters (1999) introduce a communication device that incorporates the market information generated by the competing mechanisms posted by principals. In their general construction, a mechanism for a principal requires each agent to send messages from a *universal* type space. The corresponding set of equilibrium allocations may be very large: Yamashita (2010) has been the first to show that restricting attention to a subset of such mechanisms, i.e. the *recommendation mechanisms*, is sufficient to derive a folk-theorem-like result. In a recommendation mechanism, a principal commits to post a certain direct mechanism if all but one agent recommend him to do so. Recommendation mechanisms hence allow to construct a flexible system of punishments: following a unilateral deviation of a given principal, agents can coordinate to select, amongst his opponents' decisions, those inducing the most severe punishment to the deviator. As a result, any incentive compatible allocation yielding each principal a payoff above a given threshold can be supported at equilibrium, if there are at least three agents.

The present work reconsiders the effect of communication between principals and agents on equilibrium allocations taking a more traditional mechanism design perspective. That is, we evaluate the strategic role of a principal privately communicating with agents in the spirit of the canonical construction of Myerson (1982). The above-mentioned approaches to competing mechanisms disregard this possibility. Indeed, they restrict principals to communicate by posting *public* mechanisms, which implement decisions contingent on the *private* messages received from agents. Yet, to the extent that he cannot directly contract on his opponents' mechanisms, a single principal may in principle gain by sending private signals to agents so to correlate their behaviors with the decisions

[1] Applications include competing auctions (McAfee, 1993; Peters and Severinov, 1997; Viràg, 2010), competitive search (Moen, 1997; Guerrieri et al., 2010) and competition in financial markets (Biais et al., 2000; Attar et al., 2011), among many others.

[2] This result, which has been documented in single-agent contexts by Martimort and Stole (2002) and Peters (2001), is often acknowledged as a failure of the revelation principle in games with multiple principals.

of all principals. We show that this channel of communication has relevant strategic effects.

We establish our result in the simple framework in which principals compete to attract agents under *complete information*, and each agent only takes an observable action. In such a scenario, we construct an example with two principals and three agents and explicitly characterize the set of equilibrium allocations supportable by recommendation mechanisms. In a next step, we show that *none* of the corresponding equilibria survives when all principals can send private signals to agents. By privately communicating with agents, a principal can make them differently informed of his final decisions. This uncertainty, which cannot be reproduced by standard stochastic mechanisms without signals, crucially affects the continuation game played by agents. We exploit this insight to construct a mechanism with private communication yielding a principal a payoff greater than any of those available without private communication. The result obtains despite the fact that his opponent also sends private signals and delegates to the agents the choice of the (worst) punishment against his mechanism. In the context of the example, this shows that the set of equilibrium allocations supportable by mechanisms with private signals for principals and the set of those supported by mechanisms which do not involve such private communication are disjoint. Finally, we characterize an equilibrium allocation supported by mechanisms with signals, which shows that this enlarged game admits an equilibrium. Yet, equilibrium allocations are typically not unique as we shortly discuss.

A direct implication of our main result is that the equilibria characterized by allowing only agents to privately communicate through possibly large message spaces, as in Epstein and Peters (1999), may not be robust against unilateral deviations towards mechanisms featuring principals' private communication. This in turn indicates that such signals may need to be included in any canonical system of communication, which calls for more theoretical work to identify a corresponding canonical set of equilibrium mechanisms.

To the extent that agents' observable actions can naturally be interpreted as participation decisions, the setting of the example is common to a large number of applications of competing mechanism models in which agents' participation decisions are strategic.[3] Alternatively, our example can be reconciled with economic models of competing mechanisms under complete information, in which agents participate with all principals and principals post incentive schemes that assign a decision to each profile of agents' observable actions. This is, for instance, the approach followed by Prat and Rustichini (2003) to model the lobbying process in the presence of several policy makers.

[3] We detail this interpretation in Section 4. Observe that participation is strategic in all the applications mentioned in Footnote 1.

Under complete information, these incentive schemes are interpreted as direct mechanisms. As we discuss in Section 4, an implication of our analysis is that the restriction to such direct mechanisms is problematic once principals are allowed to design more sophisticated ones. This stands in contrast with the result of Han (2007), who establishes the robustness of equilibria supported by direct mechanisms against unilateral deviations to indirect ones in competing mechanism games of complete information. Yet, he only considers mechanisms, which allow agents to send private messages to principals but *do not* allow principals to send them private signals, a restriction that we prove to be critical.

Our analysis can be casted within the framework of Yamashita (2010) once agents' actions are taken into account. An important limitation of Yamashita (2010) is the focus on deterministic behaviors. That is, agents play pure strategies in every continuation equilibrium, and principals cannot post random contracts. Szentes (2010) shows that the latter restriction is critical for the validity of Yamashita (2010)'s main result by exhibiting equilibrium allocations supported by deterministic mechanisms that yield a principal a payoff *below* Yamashita (2010)'s relevant threshold.[4] We admit instead random contracts and mixed strategy equilibria in the agents' continuation game. In our complete information example, if principals do not privately communicate with agents, recommendation mechanisms allow to re-establish a folk-theorem result in the spirit of Yamashita (2010).

Several folk-theorem results have recently been established in the competing mechanism literature. Generalizing the approach of Yamashita (2010), Peters and Troncoso-Valverde (2013) construct an abstract framework in which all players have commitment power and (privately) communicate with each other. The equilibrium distributions over players' decisions can also be correlated, due to the presence of a public correlating device. Under complete information, they show that all the allocations characterized by Yamashita (2010) are supported at equilibrium, together with those arising due to (public) correlation. We consider, instead, the situation in which only a subset of players (the principals) is able to commit while the remaining ones (the agents) take actions given the mechanisms. In this context, we allow each principal to correlate his decisions to the signals he privately sends to each agent. This feature drastically affects equilibrium analysis, since none of the allocations characterized by recommendation mechanisms can now be supported at equilibrium.

A different strategy is followed by Kalai et al. (2010), Peters and Szentes (2012), Peters (2015), and Szentes (2015) who provide attempts at modeling contractible contracts. These works show

[4] See Peters (2014) for a discussion.

that by posting contracts that *directly* refer to each other, a principal may successfully deter his opponents' deviations. A folk theorem may hence obtain even if no communication takes place after mechanisms are posted, which limits the strategic role of agents and the power of the private communication we exploit.

The feature that principals can send private signals to agents is also key in the literature on information design with multiple senders in which signals affect agents' posterior probabilities over an unknown state of the world. Kamenica and Gentzkow (2017a,b) consider a Bayesian persuasion game with a single receiver in which each sender's set of signals is sufficiently large to include signals that are effectively correlated with those of the other senders. Koessler et al. (2018) extend this approach in several directions, including the presence of multiple receivers, and focus on uncorrelated signals. We take a more traditional mechanism design perspective in which principals do not hold any private information and send signals to affect agents' beliefs over their realized decisions, which induces correlated outcomes at equilibrium. Our results hold for arbitrarily rich sets of signals available to principals.

This paper is organized as follows: Section 2 introduces a general competing mechanism model, Section 3 presents our example, Section 4 provides a discussion, and Section 5 concludes.

2 The model

We study extensive form games of complete information in which $J \geq 2$ principals deal with $I \geq 2$ agents. Each agent $i = 1, 2, \cdots, I$ takes an action a^i from a finite set A^i, and we denote $a = \left(a^1, \ldots, a^I\right) \in A = \bigtimes_{i=1}^{I} A^i$. Let Y_j be the finite set of decisions available to principal j with generic element $y_j \in Y_j$, and $Y = \bigtimes_{j=1}^{J} Y_j$. The payoff functions of agent i and of principal j are given by $u^i : A \times Y \to \mathbb{R}$ and $v_j : A \times Y \to \mathbb{R}$, respectively.

Agents' actions are observable, so each principal j can choose a decision y_j contingent on the array a. We denote $\alpha_j : A^1 \times ... \times A^I \longrightarrow \Delta(Y_j)$ an incentive scheme for principal j, with $\Delta(Y_j)$ being the set of probability distributions over Y_j. An incentive scheme specifies a (possibly stochastic) decision for every array of observed actions. We let $\mathcal{Y}_j$ be the set of incentive schemes for principal j, with $\alpha_j \in \mathcal{Y}_j$ and $\mathcal{Y} = \bigtimes_{j=1}^{J} \mathcal{Y}_j$.

2.1 Competing mechanism games: equilibrium

We first introduce the standard approach to model communication in competing mechanisms games of complete information, absent any moral hazard.[5] In this framework, communication takes place via the *private* messages sent by agents to principals, and via the *public* mechanisms principals commit to. Specifically, we let $m_j^i \in M_j^i$ be a message privately sent by agent i to principal j. A mechanism for principal j is the mapping $\gamma_j : M_j \to \mathcal{Y}_j$, in which $M_j = \underset{i=1}{\overset{I}{\times}} M_j^i$ is the set of message profiles that principal j receives from agents, with typical element $m_j = \left(m_j^1, \ldots, m_j^I\right)$. We denote $\Gamma_j^{M_j}$ the set of mechanisms available to principal j, and let $\Gamma^M = \underset{j=1}{\overset{J}{\times}} \Gamma_j^{M_j}$. If each M_j^i set is a singleton, then γ_j corresponds to an incentive scheme α_j. In this complete information setting, any such α_j is also referred to as a *direct* mechanism for principal j.

The competing mechanism game unfolds as follows. First, principals simultaneously post mechanisms. Then, agents simultaneously take their communication decisions, which determine a profile of incentive schemes $(\alpha_1, ..., \alpha_J)$. Given the public mechanisms and the messages she sent to principals, each agent takes an action, and payoffs are determined. We let $\mu^i : \Gamma^M \to \Delta\left(M^i\right)$ be the message strategy of agent i, with $M^i = \underset{j=1}{\overset{J}{\times}} M_j^i$, and $\eta^i : \Gamma^M \times M^i \to \Delta\left(A^i\right)$ be her action strategy. We take $\beta^i = (\mu^i, \eta^i)$ to be a strategy for agent i, and $\beta = (\beta^1, \ldots, \beta^I)$ a profile of strategies. A pure strategy for principal j is a mechanism $\gamma_j \in \Gamma_j^{M_j}$. We let $U^i(\gamma_j, \gamma_{-j}, \beta)$ and $V_j(\gamma_j, \gamma_{-j}, \beta)$ be the corresponding expected utilities for agent i and principal j, respectively. We denote G^M the game in which agents send messages to principals through the sets $(M^1, ..., M^I)$ and principals post mechanisms $\gamma = (\gamma_j, \gamma_{-j}) \in \Gamma^M$. We consider the subgame perfect Nash equilibria (SPNE) of G^M in which principals play pure strategies. The agents' strategies $\beta = (\beta^i, \beta^{-i})$ constitute a continuation equilibrium relative to Γ^M if, for every i and for every $\gamma \in \Gamma^M$, β^i maximizes $U^i\left(\gamma, \beta^i, \beta^{-i}\right)$ given β^{-i}. The strategies (γ, β) constitute a SPNE in G^M if β is a continuation equilibrium and if, given γ_{-j} and β, for every $j = 1, \ldots, J$: $\gamma_j \in \underset{\gamma_j' \in \Gamma_j^{M_j}}{argmax}\ V_j(\gamma_j', \gamma_{-j}, \beta)$. That is, at the stage of designing his mechanism, each principal must anticipate the Nash equilibrium of the agents' game induced by the whole array of principals' mechanisms.

As first documented by McAfee (1993) and Peck (1997), the set of equilibrium allocations of such games is crucially affected by the characteristics of the message spaces $(M^1, ..., M^I)$. Letting agents communicate, on top of their (exogenous) private information, the *market* information generated by the presence of several competing mechanisms allows principals to implement additional threats,

[5] We follow Epstein and Peters (1999), Peters (2001) and Han (2007).

thereby supporting additional allocations at equilibrium. Epstein and Peters (1999) construct the (universal) message spaces that embed this market information. Importantly, the punishments implemented using such sophisticated agents' reports against a deviating principal can be replicated by focusing on a simpler class of mechanisms.[6] These are the *recommendation* mechanisms exhibited in Yamashita (2010). To properly describe them, let $\mathcal{Y}_j \subseteq M_j^i$ for each i and j. That is, let the message spaces be sufficiently rich to allow every agent to communicate a direct mechanism to each principal j. Then, γ_j^R is a recommendation mechanism for principal j if:

$$\gamma_j^R(m_j^1, \ldots, m_j^I) = \begin{cases} \alpha_j & \text{if } \left|\left\{i : m_j^i = \alpha_j\right\}\right| \geq I - 1 \\ \text{any } \bar{\alpha}_j \in \mathcal{Y}_j & \text{otherwise.} \end{cases} \tag{1}$$

A recommendation mechanism can be understood as having agents suggest to a principal the direct mechanism to be implemented, and having the principal commit to follow any such recommendation if it is sent by at least $I - 1$ agents.

2.2 Principals' private communication: equilibrium and robustness

We now extend the construction above to cope with principals' private communication. In principle, there are many ways to enrich communication and incorporate this additional channel. Along the lines of Myerson (1982), we consider the simple case, in which each principal j sends a private signal $s_j^i \in S_j^i$ to each agent i after having received agents' messages $m_j \in M_j$. Our aim is to evaluate whether the equilibrium allocations of a given game G^M survive in enlarged games in which principals can also privately communicate to agents.

A *mechanism with signals* for principal j is the mapping $\hat{\gamma}_j : M_j \to \Delta(\mathcal{Y}_j \times S_j)$, in which $S_j = \underset{i=1}{\overset{I}{\times}} S_j^i$ is the set of signals available to principal j. Thus, given the messages m_j he receives, $\hat{\gamma}_j$ determines a joint probability distribution over principal j's incentive schemes in $\mathcal{Y}_j$ and signals in S_j. As in Myerson (1982), each agent i privately observes the realization of each signal s_j^i, and revises her prior information accordingly. Since a mechanism with signals for principal j cannot be made contingent on his opponents' mechanisms, agent i constructs her posteriors over principal j's decisions only relying on the private signal s_j^i she gets from him. We take $\Gamma_j^{M_j S_j}$ to be the set of mechanisms with signals available to principal j, and denote $\Gamma^{MS} = \underset{j=1}{\overset{J}{\times}} \Gamma_j^{M_j S_j}$.

Mechanisms with signals are publicly observed, but the message from agent i to principal j and the signal from principal j to agent i are only observed by i and j. Since signals are

[6] The formal argument is provided in Lemma 2 of Yamashita (2010).

private, a principal can generate incomplete information among agents at the stage in which they choose actions. We denote G^{MS} the extensive form game in which principals post mechanisms $\hat{\gamma} \in \Gamma^{MS}$, receive messages from agents through the sets $(M^1, ..., M^I)$, and send signals through the sets $(S_1, ..., S_J)$. As in any G^M game, there are two stages in which agent i moves in a G^{MS} game. First, having observed the mechanisms $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_J)$, she sends an array of messages $m^i = (m^i_1, \ldots, m^i_J)$ to the principals. Second, having observed her private signals $s^i = (s^i_1, \ldots, s^i_J)$, she chooses an action a^i. We take $\hat{\mu}^i : \Gamma^{MS} \to \Delta(M^i)$ to be the message strategy of agent i and $\hat{\eta}^i : \Gamma^{MS} \times M^i \times S^i \to \Delta(A^i)$ to be her strategy in the action game, with $S^i = \underset{j=1}{\overset{J}{\times}} S^i_j$. We let $\hat{\beta}^i = (\hat{\mu}^i, \hat{\eta}^i)$ be a strategy for agent i, and we extend the notion of continuation equilibrium given in Section 3.1, accordingly. For a given profile of mechanisms, agents' messages, and realized signals, we hence consider the Nash equilibria of the induced action game. Since, in any G^{MS} game, each principal may independently correlate his signals with his decisions, the equilibrium distributions of players' decisions will typically not be independent.

If there is only one principal, i.e. $J = 1$, a game G^{MS} corresponds to a complete information version of the generalized principal-agent problems analyzed in Myerson (1982).[7] In that spirit, we refer to a *direct mechanism with signals* as to a mechanism in which a principal does not ask for any message and privately signals to each agent an action to take. Formally, we denote $\hat{\hat{\gamma}}_j \in \Delta(A \times \mathcal{Y}_j)$ a direct mechanism with signals and $\hat{\hat{\Gamma}}_j \subseteq \Gamma^{M_j S_j}_j$ the set of such mechanisms for principal j.

One should observe that, for each $(M^1, ..., M^I)$, the corresponding game G^M can be interpreted as a degenerate game G^{MS} in which each S^i_j set is a singleton. In particular, we can write $\Gamma^{M_j}_j \subseteq \Gamma^{M_j S_j}_j$ for each j and S_j, and specify any mechanism without signals γ_j as a degenerate mechanism with signals $\hat{\gamma}_j$ in which, for every pair (m_j, a), the probability distribution over Y_j coincides with $\gamma_j(m_j, a)$ for each $s^i_j \in S^i_j$.[8] Following Epstein and Peters (1999) and Peters (2001), we say that an equilibrium (γ, β) of G^M is *robust* if, when considering "larger" games in which additional mechanisms are feasible, the original equilibrium survives to any unilateral deviation of a principal toward a more sophisticated mechanism. That is, if there exists *at least one* continuation equilibrium of each of these larger games which makes the deviation unprofitable.[9]

[7]In Myerson (1982), agents may also have private information and take non-observable actions.

[8]A similar reasoning is used by Peters (2001) and Han (2007) to specify a direct mechanism as a degenerate indirect one.

[9]See Epstein and Peters (1999, p. 133-134), and Peters (2001, p. 1364) for a formal definition.

3 The role of two-sided private communication: an example

This section establishes our main result. The argument is developed by means of an example which achieves two distinct objectives. First, it characterizes the equilibrium allocations supported by recommendation mechanisms. As in the incomplete information scenario of Yamashita (2010), we get a folk-theorem like result: each incentive feasible allocation yielding each principal a payoff above a given threshold can be supported at equilibrium. Second, it shows that none of these allocations can be supported at equilibrium in any game in which all principals can use private communication.

Consider a setting with five players: two principals, $P1$ and $P2$, and three agents, $A1$, $A2$ and $A3$, who take actions in the sets $A^1 = A^2 = \{\bar{a}, \underline{a}\}$ and $A^3 = \{\bar{a}\}$. Let $P1$'s decision set be $Y_1 = \{y_{11}, y_{12}\}$, and $P2$'s one be $Y_2 = \{y_{21}, y_{22}\}$. Payoffs are represented in Table 1, in which the first two numbers in each cell denote the payoffs to $P1$ and $P2$, who respectively choose rows and columns in the outer matrix. $A1$ and $A2$, respectively, choose rows and columns in the inner matrices. The payoffs to $A1$, $A2$ and $A3$ are represented by the last three numbers in each cell.

		y_{21}			y_{22}	
		$\bar{a}$	$\underline{a}$		$\bar{a}$	$\underline{a}$
y_{11}	$\bar{a}$	$(2, 95, 10, 5, 1)$	$(2, \zeta, 3/2, 8, 1)$	$\bar{a}$	$(2, \zeta, -1/10, 0, 1)$	$(2, \zeta, -1/10, 8, 1)$
	$\underline{a}$	$(2, -1, 0, 0, 1)$	$(2, \zeta, 0, 10, 1)$	$\underline{a}$	$(2, -1, 5, 5, 1)$	$(2, \zeta, 1, -10, 1)$
		$\bar{a}$	$\underline{a}$		$\bar{a}$	$\underline{a}$
y_{12}	$\bar{a}$	$(2, 95, 10, 5, 1)$	$(2, \zeta, 3/2, 8, 1)$	$\bar{a}$	$(2, \zeta, -1, 4, 1)$	$(2, \zeta, -1, 8, 1)$
	$\underline{a}$	$(2, 5, 5, 5, 1)$	$(2, \zeta, -1, 4, 1)$	$\underline{a}$	$(2, -1, 0, 0, 1)$	$(2, \zeta, 0, -10, 1)$

Table 1: The full payoff matrix of the game

The payoffs to $P1$ and $A3$ are constantly equal to 2 and to 1 respectively, and $\zeta \le -1$ is a loss to $P2$.[10] For the sake of simplicity, we henceforth refer to the reduced matrix below, which only includes the payoffs to $P2$, $A1$ and $A2$.

3.1 No private communication for principals: feasibility and equilibrium

We first consider the situation in which principals cannot send private signals to agents. In this context, we fix agents' message sets to be sufficiently large to include the set of direct mechanisms that each principal j can post, i.e. $\mathcal{Y}_j \subseteq M^i_j$ for $i = 1, 2, 3$ and $j = 1, 2$, so that recommendation

[10] The value of ζ is used to identify the threshold for $P2$'s payoff along the lines of Yamashita (2010). See Proposition 1 and, specifically, equation (2) for its explicit characterization.

		y_{21}			y_{22}	
		$\bar{a}$	$\underline{a}$		$\bar{a}$	$\underline{a}$
y_{11}	$\bar{a}$	$(95, 10, 5)$	$(\zeta, 3/2, 8)$	$\bar{a}$	$(\zeta, -1/10, 0)$	$(\zeta, -1/10, 8)$
	$\underline{a}$	$(-1, 0, 0)$	$(\zeta, 0, 10)$	$\underline{a}$	$(-1, 5, 5)$	$(\zeta, 1, -10)$
		$\bar{a}$	$\underline{a}$		$\bar{a}$	$\underline{a}$
y_{12}	$\bar{a}$	$(95, 10, 5)$	$(\zeta, 3/2, 8)$	$\bar{a}$	$(\zeta, -1, 4)$	$(\zeta, -1, 8)$
	$\underline{a}$	$(5, 5, 5)$	$(\zeta, -1, 4)$	$\underline{a}$	$(-1, 0, 0)$	$(\zeta, 0, -10)$

Table 2: The reduced payoff matrix

mechanisms are available to both principals. In the next paragraphs, we characterize the set of allocations supported by recommendation mechanisms in an equilibrium of this G^M game.

We first identify the set of incentive feasible allocations. Since principals do not privately communicate, a direct mechanism can be conveniently represented by means of four binary distributions over principals' decisions, one for each pair of agents' actions. In what follows, we let $\pi_{a^1a^2} \equiv \mathit{prob}(y_{11}|a^1, a^2)$ be the probability with which $P1$ plays y_{11} if the actions $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ are observed. A direct mechanism for $P1$ is therefore an array $\alpha_1 = (\pi_{\bar{a}\bar{a}}, \pi_{\bar{a}\underline{a}}, \pi_{\underline{a}\bar{a}}, \pi_{\underline{a}\underline{a}}) \in [0, 1]^4$. Similarly, we let $\sigma_{a^1a^2} \equiv \mathit{prob}(y_{21}|a^1, a^2)$ be the probability with which $P2$ plays y_{21} if $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ are observed, and we write $\alpha_2 = (\sigma_{\bar{a}\bar{a}}, \sigma_{\bar{a}\underline{a}}, \sigma_{\underline{a}\bar{a}}, \sigma_{\underline{a}\underline{a}}) \in [0, 1]^4$. An (stochastic) allocation induced by the direct mechanisms (α_1, α_2) and by the strategies (η^1, η^2, η^3) is a probability distribution over final choices in $Y_1 \times Y_2 \times A^1 \times A^2 \times A^3$ defined by the array

$$z = \left((\pi_{a^1a^2})_{(a^1,a^2)\in\{\bar{a},\underline{a}\}^2}, (\sigma_{a^1a^2})_{(a^1,a^2)\in\{\bar{a},\underline{a}\}^2}, \eta^1(.|\alpha_1, \alpha_2), \eta^2(.|\alpha_1, \alpha_2), \eta^3(\bar{a}|\alpha_1, \alpha_2) = 1 \right),$$

in which $\eta^i(.|\alpha_1, \alpha_2)$ denotes the probability distribution over A^i for agent $i = 1, 2$ given (α_1, α_2). We then say that an (stochastic) allocation z is *incentive feasible* if the strategies (η^1, η^2, η^3) form an (Nash) equilibrium of the agents' action game induced by (α_1, α_2).[11] We denote Z^{IF} the set of incentive feasible allocations. The two remarks below are key for equilibrium characterization.

Remark 1 *Any allocation supported in an equilibrium of G^M is incentive feasible.*

Remark 2 *Z^{IF} is non-empty. In particular, it includes the allocation inducing the deterministic choices $(y_{12}, y_{21}, \bar{a}, \underline{a}, \bar{a})$. Indeed, if $P1$ commits to play y_{12} for each profile of agents' actions, and $P2$ makes the same commitment to y_{21}, then it is an equilibrium for A1 to play $\bar{a}$, for A2 to play $\underline{a}$*

[11] Yamashita (2010) restricts attention to *deterministic* allocations. That is, agents play pure strategies in every continuation equilibrium, and principals cannot randomize over their decisions. Under this restriction, existence of a continuation equilibrium is not guaranteed. We enlarge the analysis to random behaviors, therefore allowing for mixed strategy equilibria in each continuation game played by the agents.

(with $A3$ playing $\bar{a}$). This yields the payoffs $(2, \zeta, 3/2, 8, 1)$. A similar reasoning guarantees that Z^{IF} includes the allocation inducing the choices $(y_{11}, y_{22}, \underline{a}, \bar{a}, \bar{a})$, which yield the payoffs $(2, -1, 5, 5, 1)$. Finally, it also includes the allocation sustained by the direct mechanisms in which $P1$ commits to play y_{12} when observing the actions $(\underline{a}, \bar{a}, \bar{a})$, and y_{11} otherwise, and $P2$ commits to play y_{21} when observing the actions $(\underline{a}, \bar{a}, \bar{a})$, and y_{22} otherwise. Given these offers, $(\underline{a}, \bar{a}, \bar{a})$ is an equilibrium of the agents' action game. The induced choices are $(y_{12}, y_{21}, \underline{a}, \bar{a}, \bar{a})$, corresponding to the payoffs $(2, 5, 5, 5, 1)$.

Remark 1, which directly follows from the definition of incentive feasibility, parallels Lemma 1 in Yamashita (2010). The multiplicity of incentive feasible allocations documented in Remark 2 suggests the possibility of using recommendation mechanisms to derive a folk-theorem result in the example. This is established in the following proposition.

Proposition 1 *Every incentive feasible allocation yielding at least -1 to $P2$ can be sustained in an equilibrium of the game G^M.*

Proof. Let each principal $j = 1, 2$ use the recommendation mechanism γ_j^R as defined in (1). To develop the proof, we first establish the following lemma.

Lemma 1 *If $P1$ posts the recommendation mechanism γ_1^R then, for every mechanism $\gamma_2 \in \Gamma_2^{M_2}$ posted by $P2$, there exists an equilibrium of the agents' game yielding $P2$ at most -1.*

Proof. Let $P1$ post γ_1^R. For each $\gamma_2 \in \Gamma_2^{M_2}$ posted by $P2$, agents play a continuation game over the messages to send to principals and over their actions. Let the message profile $m_1 \in M_1$ be such that agents select in γ_1^R the direct mechanism $\alpha_1 \in \mathcal{Y}_1$ in which $\pi_{\bar{a}\bar{a}} = \pi_{\bar{a}\underline{a}} = \pi_{\underline{a}\bar{a}} = 1$ and $\pi_{\underline{a}\underline{a}} = 0$. In addition, let μ denote a probability distribution over the messages sent to $P2$ and $\sigma^\mu = (\sigma^\mu_{\bar{a}\bar{a}}, \sigma^\mu_{\bar{a}\underline{a}}, \sigma^\mu_{\underline{a}\bar{a}}, \sigma^\mu_{\underline{a}\underline{a}})$ be the profile of probability distributions over $P2$'s decisions induced by such μ, given γ_2.

Consider the agents' action game induced by the mechanisms (γ_1^R, γ_2), given the messages m_1 sent to $P1$ and the distribution μ over the messages sent to $P2$. In this game, $A3$ can only take the action $\{\bar{a}\}$, and the strategic interaction between $A1$ and $A2$ is represented in Table 3.

The game has no pure strategy equilibrium in which $A1$ and $A2$ play $(\bar{a}, \bar{a})$. Indeed, if $A1$ plays $\bar{a}$, $A2$ will choose $\underline{a}$ since $8 > 5\sigma^\mu_{\bar{a}\bar{a}}$ for every $\sigma^\mu_{\bar{a}\bar{a}} \in [0, 1]$. The following situations may hence arise.

1. The game has a pure strategy equilibrium in which $A1$ plays $\bar{a}$ and $A2$ plays $\underline{a}$, with $A3$ playing $\bar{a}$. This is for instance the case if $\sigma^\mu_{\bar{a}\underline{a}} \geq 1/16$. The equilibrium yields $P2$ the payoff $\zeta \leq -1$.

	$\bar{a}$	$\underline{a}$
$\bar{a}$	$11\sigma^{\mu}_{\bar{a}\bar{a}} + \frac{9}{10}(1-\sigma^{\mu}_{\bar{a}\bar{a}}) - 1, 5\sigma^{\mu}_{\bar{a}\bar{a}}$	$\frac{8}{5}\sigma^{\mu}_{\bar{a}\underline{a}} - \frac{1}{10}, 8$
$\underline{a}$	$5(1-\sigma^{\mu}_{\underline{a}\bar{a}}), 5(1-\sigma^{\mu}_{\underline{a}\bar{a}})$	$-\sigma^{\mu}_{\underline{a}\underline{a}}, 6\sigma^{\mu}_{\underline{a}\underline{a}} - 10$

Table 3: Agents' action game induced by (γ_1^R, γ_2) given m_1 and μ.

2. The game has a pure strategy equilibrium in which $A1$ plays $\underline{a}$ and $A2$ plays $\underline{a}$, with $A3$ playing $\bar{a}$. This is never the case since $6\sigma^{\mu}_{\underline{a}\underline{a}} - 10 < 0 \leq 5(1-\sigma^{\mu}_{\underline{a}\bar{a}})$ for every $\sigma^{\mu}_{\underline{a}\bar{a}}$ and $\sigma^{\mu}_{\underline{a}\underline{a}}$.

3. The game has a pure strategy equilibrium in which $A1$ plays $\underline{a}$ and $A2$ plays $\bar{a}$, with $A3$ playing $\bar{a}$. This is the case if $11\sigma^{\mu}_{\bar{a}\bar{a}} + 9/10(1-\sigma^{\mu}_{\bar{a}\bar{a}}) - 1 \leq 5(1-\sigma^{\mu}_{\underline{a}\bar{a}})$ which is for instance satisfied if $\sigma^{\mu}_{\bar{a}\bar{a}} = \sigma^{\mu}_{\underline{a}\bar{a}} = 0$. Since $\pi_{\underline{a}\bar{a}} = 1$, the equilibrium yields $P2$ the payoff -1.

4. The game has a *mixed* strategy equilibrium in which $A1$ plays $\bar{a}$ with probability ϕ, $A2$ plays $\bar{a}$ with probability τ, and $A3$ plays $\bar{a}$ with probability one. To have at least one player randomizing at equilibrium it must be that either

$$\frac{8}{5}\sigma^{\mu}_{\bar{a}\underline{a}} - \frac{1}{10} \geq -\sigma^{\mu}_{\underline{a}\underline{a}} \qquad \text{and} \qquad 11\sigma^{\mu}_{\bar{a}\bar{a}} + 9/10(1-\sigma^{\mu}_{\bar{a}\bar{a}}) - 1 \leq 5(1-\sigma^{\mu}_{\underline{a}\bar{a}}),$$

or

$$\frac{8}{5}\sigma^{\mu}_{\bar{a}\underline{a}} - \frac{1}{10} \leq -\sigma^{\mu}_{\underline{a}\underline{a}} \qquad \text{and} \qquad 11\sigma^{\mu}_{\bar{a}\bar{a}} + 9/10(1-\sigma^{\mu}_{\bar{a}\bar{a}}) - 1 \geq 5(1-\sigma^{\mu}_{\underline{a}\bar{a}}).$$

The expected payoff to $P2$ in a mixed strategy equilibrium is:

$$\phi\tau(95\sigma^{\mu}_{\bar{a}\bar{a}} + \zeta(1-\sigma^{\mu}_{\bar{a}\bar{a}})) - (1-\phi)\tau + (1-\tau)\zeta,$$

which is lower than -1 whenever

$$\zeta\left[\phi\tau(1-\sigma^{\mu}_{\bar{a}\bar{a}}) + (1-\tau)\right] + \tau\left[\phi\ 95\sigma^{\mu}_{\bar{a}\bar{a}} - (1-\phi)\right] \leq -1. \qquad (2)$$

The term $[\phi\tau(1-\sigma^{\mu}_{\bar{a}\bar{a}}) + (1-\tau)]$ in the left-hand side of (2) is positive and bounded away from 0 in any mixed strategy equilibrium of the action game.[12] In addition, since the term $\tau\left[\phi\ 95\sigma^{\mu}_{\bar{a}\bar{a}} - (1-\phi)\right]$ is bounded above by 95, given σ^{μ}, (2) is satisfied for every $(\phi, \tau) \in [0,1]^2$ if the loss ζ is large enough.

[12]Indeed, ϕ is bounded away from zero in any mixed strategy equilibrium, as one can verify by inspection of Table 3. For $[\phi\tau(1-\sigma^{\mu}_{\bar{a}\bar{a}}) + (1-\tau)]$ to be arbitrarily close to zero, one then needs to have $\sigma^{\mu}_{\bar{a}\bar{a}}$ converging to one and inducing an equilibrium in which τ is arbitrarily close to one. Yet, the equilibrium value of τ is decreasing in $\sigma^{\mu}_{\bar{a}\bar{a}}$, and it is bounded away from one when $\sigma^{\mu}_{\bar{a}\bar{a}}$ converges to one. Finally, observe that if $\sigma^{\mu}_{\bar{a}\bar{a}} = 1$ the agents' action game only admits a pure strategy equilibrium, in which $\phi = 1$ and $\tau = 0$, and $P2$'s payoff is exactly equal to ζ.

We therefore set $\zeta = min\{-1, \bar{\zeta}\}$. This guarantees that $P2$ cannot achieve a payoff above -1 in any equilibrium of the action game induced by a deviation to any mechanism $\gamma_2 \in \Gamma_2^{M_2}$, if agents send messages to $P1$ selecting $\pi_{\bar{a}\bar{a}} = \pi_{\underline{a}\bar{a}} = \pi_{\bar{a}\underline{a}} = 1$, and $\pi_{\underline{a}\underline{a}} = 0$, and choose the distribution $\mu \in \Delta(M_2)$ to communicate with him.

To complete of the proof of Lemma 1, we argue that, for every γ_2, there exists an equilibrium of the continuation game induced by (γ_1^R, γ_2), in which agents send the message profile m_1 to $P1$, recommending to select the direct mechanism $\alpha_1 = (1, 1, 1, 0)$. That these behaviors are part of an equilibrium is indeed a direct implication of $P1$ posting a recommendation mechanism in the presence of three agents, which guarantees that the majority rule in (1) applies. ■

To complete the proof of Proposition 1, we specify the agents' equilibrium strategies in such a way that, following each deviation of $P2$, they recommend $\alpha_1 = (1, 1, 1, 0)$ to $P1$ and coordinate on a profile of actions yielding $P2$ a payoff of at most -1. ■

The above reasoning reproduces that of Lemma 2 in Yamashita (2010). We argue that the payoff -1 is the minmax value for $P2$ over incentive schemes taking into account the subsequent action game played by agents. Indeed, for each direct mechanism posted by $P1$, $P2$ can always guarantee himself the payoff -1, as clarified in the following remark.

Remark 3 *Take any $\alpha_1 = (\pi_{\bar{a}\bar{a}}, \pi_{\bar{a}\underline{a}}, \pi_{\underline{a}\bar{a}}, \pi_{\underline{a}\underline{a}}) \in [0, 1]^4$, and let $P2$ post a direct mechanism α_2 such that $\sigma_{\bar{a}\bar{a}} = \sigma_{\bar{a}\underline{a}} = \sigma_{\underline{a}\underline{a}} = 0$. Then, $(\underline{a}, \bar{a})$ is the only equilibrium of the agents' action game. That is, the game induced by the direct mechanisms (α_1, α_2), has an equilibrium yielding -1 to $P2$. Thus, there is no direct mechanism for $P1$ which allows to punish $P2$ with a payoff below -1. In addition, as shown in the proof of Lemma 1, there is an α_1 which prevents $P2$ from achieving a payoff above -1 for every direct mechanism α_2 she may choose. These observations guarantee that the minmax payoff value for $P2$ is exactly -1.*

The value -1 also corresponds to the minimal equilibrium payoff for $P2$ in a complete information game in which each principal posts recommendation mechanisms and agents take actions and coordinate on the worst continuation equilibrium for $P2$, in analogy with the threshold identified by Yamashita (2010).[13]

Key to our analysis is to characterize the *maximal* payoff that $P2$ can attain at equilibrium if he cannot privately communicate with agents. Given Proposition 1, this corresponds to his maximal payoff computed over the set Z^{IF} and it is characterized in the following lemma.

[13]See Peters (2014) for a general discussion of the minmax characterized by Yamashita (2010) in terms of the primitives of a competing mechanism game.

Lemma 2 *The maximal payoff to $P2$ over all allocations $z \in Z^{IF}$ is 5.*

Proof. Table 4 below depicts the action game played by $A1$ and $A2$ for a given profile of direct mechanisms, $\alpha_1 = (\pi_{\bar{a}\bar{a}}, \pi_{\bar{a}\underline{a}}, \pi_{\underline{a}\bar{a}}, \pi_{\underline{a}\underline{a}})$ and $\alpha_2 = (\sigma_{\bar{a}\bar{a}}, \sigma_{\bar{a}\underline{a}}, \sigma_{\underline{a}\bar{a}}, \sigma_{\underline{a}\underline{a}})$, recalling that $A3$ can only play $\{\bar{a}\}$.

	$\bar{a}$	$\underline{a}$
$\bar{a}$	$11\sigma_{\bar{a}\bar{a}} + \frac{9}{10}\pi_{\bar{a}\bar{a}}(1-\sigma_{\bar{a}\bar{a}}) - 1,$ $\sigma_{\bar{a}\bar{a}} + 4(1 - \pi_{\bar{a}\bar{a}} + \pi_{\bar{a}\bar{a}}\sigma_{\bar{a}\bar{a}})$	$\frac{5}{2}\sigma_{\bar{a}\underline{a}} + \frac{9}{10}\pi_{\bar{a}\underline{a}}(1-\sigma_{\bar{a}\underline{a}}) - 1, 8$
$\underline{a}$	$5(\sigma_{\underline{a}\bar{a}} + \pi_{\underline{a}\bar{a}}) - 10\sigma_{\underline{a}\bar{a}}\pi_{\underline{a}\bar{a}},$ $5(\sigma_{\underline{a}\bar{a}} + \pi_{\underline{a}\bar{a}}) - 10\sigma_{\underline{a}\bar{a}}\pi_{\underline{a}\bar{a}}$	$\pi_{\underline{a}\underline{a}} - \sigma_{\underline{a}\underline{a}},$ $\sigma_{\underline{a}\underline{a}}(6\pi_{\underline{a}\underline{a}} + 10) - 6$

Table 4: The actions' game played by $A1$ and $A2$, induced by (α_1, α_2)

As pointed out in Remark 2, there exists an allocation $z \in Z^{IF}$ yielding 5 to $P2$. For $P2$ to achieve a payoff strictly above 5, principals' mechanisms should be designed to induce agents to choose $(\bar{a}, \bar{a})$ with positive probability. Yet, in any equilibrium of the above game in which at least one agent randomizes, the payoff to $P2$ is smaller than 5. That is:

$$\begin{aligned}\phi\tau(95\sigma_{\bar{a}\bar{a}} + \zeta(1-\sigma_{\bar{a}\bar{a}})) + (1-\phi)\tau[6\sigma_{\underline{a}\bar{a}}(1-\pi_{\underline{a}\bar{a}}) - 1] + (1-\tau)\zeta = \\ \zeta\left[\phi\tau(1-\sigma_{\bar{a}\bar{a}}) + (1-\tau)\right] + \tau\left[\phi\ 95\sigma_{\bar{a}\bar{a}} - (1-\phi)\right] + \tau(1-\phi)6\sigma_{\underline{a}\bar{a}}(1-\pi_{\underline{a}\bar{a}}) \leq 5\end{aligned} \tag{3}$$

for every mixed strategy equilibrium (ϕ, τ) induced by any (α_1, α_2). To establish the inequality in (3), recall that, by (2), $\zeta\left[\phi\tau(1-\sigma_{\bar{a}\bar{a}}) + (1-\tau)\right] + \tau\left[\phi\ 95\sigma_{\bar{a}\bar{a}} - (1-\phi)\right] \leq -1$ in any mixed strategy equilibrium. It follows that:

$$\zeta\left[\phi\tau(1-\sigma_{\bar{a}\bar{a}}) + (1-\tau)\right] + \tau\left[\phi\ 95\sigma_{\bar{a}\bar{a}} - (1-\phi)\right] + \tau(1-\phi)6\sigma_{\underline{a}\bar{a}}(1-\pi_{\underline{a}\bar{a}}) \leq \tau(1-\phi)6\sigma_{\underline{a}\bar{a}}(1-\pi_{\underline{a}\bar{a}}) - 1 \leq 5$$

holds for every (α_1, α_2). To conclude the proof it remains to show that $(\bar{a}, \bar{a}, \bar{a})$ cannot be an (pure strategy) equilibrium of the agents' action game. Indeed, since $\sigma_{\bar{a}\bar{a}} + 4(1 - \pi_{\bar{a}\bar{a}} + \pi_{\bar{a}\bar{a}}\sigma_{\bar{a}\bar{a}}) < 8$ for each $(\sigma_{\bar{a}\bar{a}}, \pi_{\bar{a}\bar{a}})$, if $A1$ plays $\bar{a}$, $A2$ strictly prefers to play $\underline{a}$. Hence, there is no $z \in Z^{IF}$ yielding $P2$ a payoff strictly greater than 5. ■

One should observe that, to achieve his maximal payoff, $P2$ crucially exploits the possibility to contract on agents' observable actions (see Remark 2). If principals' decisions were not contingent on agents' actions, then there would not be a feasible allocation yielding $P2$ the (maximal) payoff of 5. Indeed, any such allocation would necessarily involve $P1$ playing y_{12} and $P2$ playing y_{21} with probability one, $A1$ and $A2$ playing the pure strategies $(\underline{a}, \bar{a})$. One can then check that, given these principals' decisions, $(\underline{a}, \bar{a})$ would *not* be an equilibrium of the agents' action game.

Taken together, Proposition 1 and Lemma 2 imply that recommendation mechanisms support all incentive feasible allocations yielding a payoff above -1 and at most equal to 5 to $P2$ in an equilibrium of the above game G^M. This provides an instance of Yamashita (2010)'s Theorem 1 in a complete information setting in which random behaviors are allowed.[14] We remark that the lower bound of $P2$'s payoff coincides with -1 in any G^M in which the message sets of $P1$ are sufficiently rich to include all his direct mechanisms. The upper bound, instead, is equal to 5 regardless of the size of any principal's message sets, as the proof of Lemma 2 shows.

3.2 Principals' private communication: equilibrium analysis

We now consider the situation in which each principal j posts a mechanism with signals $\hat{\gamma}_j \in \Gamma_j^{M_j S_j}$, recalling that $\Gamma_j^{M_j} \subseteq \Gamma_j^{M_j S_j}$. In such enlarged setting, we show that for every mechanism with signals posted by $P1$, there is a mechanism with signals yielding $P2$ a payoff strictly greater than 5. Hence, none of the allocations characterized in Proposition 1 can be supported at equilibrium. That is, that the set of equilibrium allocations of any game G^{MS} and the set of those of the corresponding game G^M are disjoint. The result is established in the following proposition.

Proposition 2 *Consider a game G^{MS}, in which S_j^i is a finite set and $A^i \subseteq S_j^i$ for every (i, j). Let $P1$ post an arbitrary mechanism $\hat{\gamma}_1 \in \Gamma_1^{M_1 S_1}$. Then, there exists $\hat{\gamma}_2 \in \Gamma_2^{M_2 S_2}$ which yields $P2$ a payoff strictly greater than 5 in every continuation equilibrium.*

Proof. The proof shows that $P2$ can always attain a payoff greater than 5 by means of a simple mechanism, in which he sends to each agent a private signal on the action she should take and he commits to a joint probability distribution over signals and incentive schemes that is not contingent on agents' messages. Therefore, $\hat{\gamma}_2$ is a direct mechanism with signals. Specifically, it prescribes that:

i.) $P2$ privately communicates $\{\bar{a}\}$ to all agents and chooses y_{21} for every profile of agents' actions, with probability $k > 0$;

ii.) $P2$ privately communicates $\{\underline{a}\}$ to $A1$ and $\{\bar{a}\}$ to $A2$ and $A3$ and chooses y_{22} for every profile of agents' actions, with probability $(1 - k)$.

[14] See also Xiong (2013) for a version of the folk theorem of Yamashita (2010) that does not rely on the restriction to deterministic behaviors.

The mechanism $\hat{\gamma}_2$ implements the above distribution for every profile of agents' messages received by $P2$. Given the signal she privately receives from $P2$, each agent i is able to construct the conditional joint probability over $\{y_{21}, y_{22}\}$ and signals sent by $P2$ to her opponents. In particular, $\hat{\gamma}_2$ is such that, given her private signal, $A1$ knows exactly which decision $P2$ is implementing, while $A2$ remains uninformed. We let $q_2^1(y_{21}, \bar{a}|\bar{a})$ be the conditional probability formed by $A1$ on $P2$ choosing y_{21} and signaling $\bar{a}$ to $A2$, when she receives $\bar{a}$ from him.[15] Observe that given $\hat{\gamma}_2$, one has $q_2^1(y_{21}, \bar{a}|\bar{a}) = 1$ for $A1$. Similarly, we let $q_2^1(y_{22}, \bar{a}|\underline{a})$ be the conditional probability formed by $A1$ on $P2$ choosing y_{22} and signaling $\bar{a}$ to $A2$, when she gets $\underline{a}$ from him. This is also equal to 1 when $P2$ commits to $\hat{\gamma}_2$. All other posteriors probabilities for $A1$ are null given i.)-ii.).

On the contrary, $A2$ only receives the signal $\bar{a}$ with positive probability in $\hat{\gamma}_2$, which implies that her posteriors are equal to the priors, i.e. $q_2^2(y_{21}, \bar{a}|\bar{a}) = k$ and $q_2^2(y_{22}, \underline{a}|\bar{a}) = 1 - k$.

We now show that $\hat{\gamma}_2$ yields $P2$ a payoff greater than 5, for every mechanism $\hat{\gamma}_1 \in \Gamma_1^{M_1 S_1}$ posted by $P1$. To do so, we have to consider $P1$'s probability distribution over incentive schemes and signals as determined by the messages that agents send him in the game induced by $(\hat{\gamma}_1, \hat{\gamma}_2)$. We denote this joint probability $q_1 \in \Delta(\mathcal{Y}_1 \times S_1)$, with $S_1 = S_1^1 \times S_1^2 \times S_1^3$.

Given the (private) signal received from $P1$, each agent $i = 1, 2, 3$ constructs the conditional probabilities over incentive schemes in $\alpha_1 \in \mathcal{Y}_1$ and signals to her opponents $s_1^{-i} \in S_1^{-i}$. Specifically, we let $q_1^i(\alpha_1, s_1^{-i}|s_1^i)$ be the conditional probability that agent i assigns to $P1$ choosing the incentive scheme α_1 and signaling the array s_1^{-i} to her opponents, when she receives the signal $s_1^i \in S_1^i$.

We develop the argument in two steps. First, we consider distributions in which $P1$ directly signals an action to each agent, that is $q_1 \in \Delta(\mathcal{Y}_1 \times A^1 \times A^2)$, recalling that $A3$ takes only one action. Then, we extend the proof to the general case in which $P1$ uses arbitrary signals in S_1^i for every $i = 1, 2$.[16]

Step 1. Given $(\hat{\gamma}_1, \hat{\gamma}_2)$, let the agents' messages select a $q_1 \in \Delta(\mathcal{Y}_1 \times A^1 \times A^2)$. Then, $q_1^i(\alpha_1, a^j|a^i)$ is the conditional probability that agent $i = 1, 2$ assigns to $P1$ choosing the incentive scheme α_1 and signaling $a^j \in \{\bar{a}, \underline{a}\}$ to agent $j \neq i$, when she receives the signal $a^i \in \{\bar{a}, \underline{a}\}$ from $P1$. In addition, we denote $\pi_{a^1 a^2}^{\alpha_1}$ the probability that the incentive scheme α_1 assigns to y_{11} given the agents' actions $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$.

Given $(\hat{\gamma}_1, \hat{\gamma}_2)$, we henceforth refer to the agents' action game induced by any profile of messages which select q_1. In this game, agents take actions given the realization of principals' private signals.

[15] As clarified in Section 3.2, the private signal that agent i receives from principal j is the only relevant information to construct her posterior probabilities on principal j's decisions.

[16] There is no loss of generality in assuming that S_1^3 is a singleton. Indeed, the private signals sent to an agent affect her opponents' payoffs only to the extent that they effectively modify her actions.

We show that playing in accordance with the signal she gets from $P2$ is a dominant strategy for $A1$. That is, she strictly prefers to follow $P2$'s private signal for every *pure* strategy chosen by $A2$ in the action game. To do so, we consider the four cases corresponding to the possible combinations of principals' signals she may receive.

1.) $A1$ receives the signal $\bar{a}$ from both principals. Given these signals, and since $q_2^1(y_{21}, \bar{a}|\bar{a}) = 1$, her expected payoff when choosing $a^1 \in \{\bar{a}, \underline{a}\}$ against the pure action strategy $\hat{\eta}^2$ of her opponent, is:[17]

$$\int_{\alpha_1} q_1^1(\alpha_1, \bar{a}|\bar{a}))[\pi^{\alpha_1}_{a^1\hat{\eta}^2(\bar{a})}\, u^1(y_{11}, y_{21}, a^1, \hat{\eta}^2(\bar{a})) + (1 - \pi^{\alpha_1}_{a^1\hat{\eta}^2(\bar{a})})\, u^1(y_{12}, y_{21}, a^1, \hat{\eta}^2(\bar{a}))]d\alpha_1$$
$$+ \int_{\alpha_1} q_1^1(\alpha_1, \underline{a}|\bar{a})[\pi^{\alpha_1}_{a^1\hat{\eta}^2(\underline{a})}\, u^1(y_{11}, y_{21}, a^1, \hat{\eta}^2(\underline{a})) + (1 - \pi^{\alpha_1}_{a^1\hat{\eta}^2(\underline{a})})\, u^1(y_{12}, y_{21}, a^1, \hat{\eta}^2(\underline{a}))]d\alpha_1, \tag{4}$$

in which, with some abuse of notation, we let $\hat{\eta}^2(s) \in \{\bar{a}, \underline{a}\}$ be the action that the strategy $\hat{\eta}_2$ prescribes to $A2$ when receiving the signal $s \in \{\bar{a}, \underline{a}\}$ from $P1$. We now determine $A1$'s optimal actions given her beliefs on $A2$'s behavior, which leads to consider the following four sub-cases.

1a.) $\hat{\eta}^2$ prescribes to $A2$ to play $\bar{a}$ for every signal she receives from $P1$, i.e. $\hat{\eta}^2(\bar{a}) = \hat{\eta}^2(\underline{a}) = \bar{a}$. In this case, one can check from Table 2 that $A1$ gets $\int_{\alpha_1} 10\left(q_1^1(\alpha_1, \bar{a}|\bar{a}) + q_1^1(\alpha_1, \underline{a}|\bar{a})\right) d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1} 5(1 - \pi^{\alpha_1}_{\underline{a}\bar{a}})\left(q_1^1(\alpha_1, \bar{a}|\bar{a}) + q_1^1(\alpha_1, \underline{a}|\bar{a})\right) d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{\underline{a}\bar{a}} \in [0, 1]$ for each α_1,

$$\int_{\alpha_1} 10\left(q_1^1(\alpha_1, \bar{a}|\bar{a}) + q_1^1(\alpha_1, \underline{a}|\bar{a})\right) d\alpha_1 > \int_{\alpha_1} 5(1 - \pi^{\alpha_1}_{\underline{a}\bar{a}})\left(q_1^1(\alpha_1, \bar{a}|\bar{a}) + q_1^1(\alpha_1, \underline{a}|\bar{a})\right) d\alpha_1,$$

hence, $A1$ strictly prefers $\bar{a}$ to $\underline{a}$ for every $q_1^1(\alpha_1, \underline{a}|\bar{a}), q_1^1(\alpha_1, \bar{a}|\bar{a})$ and $\pi^{\alpha_1}_{\underline{a}\bar{a}}$.

1b.) $\hat{\eta}^2$ prescribes to $A2$ to play $\bar{a}$ ($\underline{a}$) if she gets the signal $\bar{a}$ ($\underline{a}$) from $P1$, i.e. $\hat{\eta}^2(\bar{a}) = \bar{a}$, $\hat{\eta}^2(\underline{a}) = \underline{a}$. In this case, $A1$ gets $\int_{\alpha_1}\left[10q_1^1(\alpha_1, \bar{a}|\bar{a}) + \frac{3}{2}q_1^1(\alpha_1, \underline{a}|\bar{a})\right] d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1}\left[5(1 - \pi^{\alpha_1}_{\underline{a}\bar{a}})q_1^1(\alpha_1, \bar{a}|\bar{a}) - (1 - \pi^{\alpha_1}_{\underline{a}\underline{a}})q_1^1(\alpha_1, \underline{a}|\bar{a})\right] d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{\underline{a}\bar{a}}$ and $\pi^{\alpha_1}_{\underline{a}\underline{a}}$ are smaller than one for each α_1,

$$\int_{\alpha_1}\left[10q_1^1(\alpha_1, \bar{a}|\bar{a}) + \frac{3}{2}q_1^1(\alpha_1, \underline{a}|\bar{a})\right] d\alpha_1 > \int_{\alpha_1}\left[5(1 - \pi^{\alpha_1}_{\underline{a}\bar{a}})q_1^1(\alpha_1, \bar{a}|\bar{a}) - (1 - \pi^{\alpha_1}_{\underline{a}\underline{a}})q_1^1(\alpha_1, \underline{a}|\bar{a})\right] d\alpha_1,$$

and $A1$ strictly prefers $\bar{a}$ to $\underline{a}$ for every $q_1^1(\alpha_1, \underline{a}|\bar{a}), q_1^1(\alpha_1, \bar{a}|\bar{a})$ and $(\pi^{\alpha_1}_{\underline{a}\bar{a}}, \pi^{\alpha_1}_{\underline{a}\underline{a}})$.

1c.) $\hat{\eta}^2$ prescribes to $A2$ to play $\bar{a}$ ($\underline{a}$) if she gets the signal $\underline{a}$ ($\bar{a}$) from $P1$, i.e. $\hat{\eta}^2(\bar{a}) = \underline{a}$, $\hat{\eta}^2(\underline{a}) = \bar{a}$. In this case, $A1$ gets $\int_{\alpha_1}\left[\frac{3}{2}q_1^1(\alpha_1, \bar{a}|\bar{a}) + 10q_1^1(\alpha_1, \underline{a}|\bar{a})\right] d\alpha_1$ by playing $\bar{a}$, and she would

[17] To simplify notation, throughout the proof we deliberately omit to specify the action $\{\bar{a}\}$ taken by $A3$ in the expressions of $A1$'s and $A2$'s expected payoffs.

get $\int_{\alpha_1}\left[-(1-\pi^{\alpha_1}_{\underline{a}\underline{a}})q^1_1(\alpha_1,\bar{a}|\bar{a})+5(1-\pi^{\alpha_1}_{\underline{a}\bar{a}})q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{\underline{a}\bar{a}}$ and $\pi^{\alpha_1}_{\underline{a}\underline{a}}$ are smaller than one for each α_1,

$$\int_{\alpha_1}\left[3/2q^1_1(\alpha_1,\bar{a}|\bar{a})+10q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1>\int_{\alpha_1}\left[-(1-\pi^{\alpha_1}_{\underline{a}\underline{a}})q^1_1(\alpha_1,\bar{a}|\bar{a})+5(1-\pi^{\alpha_1}_{\underline{a}\bar{a}})q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1,$$

which leads to the same conclusion of 1b.).

1d.) $\hat{\eta}^2$ prescribes to $A2$ to play $\underline{a}$ for every signal she receives from $P1$, i.e. $\hat{\eta}^2(\bar{a})=\hat{\eta}^2(\underline{a})=\underline{a}$. In this case, $A1$ gets $\int_{\alpha_1}3/2\left[q^1_1(\alpha_1,\bar{a}|\bar{a})+q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1$ by playing $\bar{a}$, and she would get $-\int_{\alpha_1}(1-\pi^{\alpha_1}_{\underline{a}\underline{a}})\left[q^1_1(\alpha_1,\bar{a}|\bar{a})+q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{\underline{a}\underline{a}}$ is smaller than one for each α_1,

$$\int_{\alpha_1}3/2\left[q^1_1(\alpha_1,\bar{a}|\bar{a})+q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1>-\int_{\alpha_1}(1-\pi^{\alpha_1}_{\underline{a}\underline{a}})\left[q^1_1(\alpha_1,\bar{a}|\bar{a})+q^1_1(\alpha_1,\underline{a}|\bar{a})\right]d\alpha_1,$$

and $A1$ strictly prefers $\bar{a}$ to $\underline{a}$ for every $q^1_1(\alpha_1,\underline{a}|\bar{a}),q^1_1(\alpha_1,\bar{a}|\bar{a})$ and $\pi^{\alpha_1}_{\underline{a}\underline{a}}$.

To resume, upon getting $(\bar{a},\bar{a})$ from both principals, it is optimal for $A1$ to play $\bar{a}$ for every $A2$'s pure action strategy $\hat{\eta}^2$.

2.) $A1$ receives the signal $\underline{a}$ from $P1$ and the signal $\bar{a}$ from $P2$. In this case, her expected payoff can be derived from (4) by substituting $(q^1_1(\alpha_1,\underline{a}|\bar{a}),q^1_1(\alpha_1,\bar{a}|\bar{a}))$ with $(q^1_1(\alpha_1,\underline{a}|\underline{a}),q^1_1(\alpha_1,\bar{a}|\underline{a}))$. As a consequence, to determine $A1$'s optimal actions one can follow the analysis developed in 1a.)-1d.), which leads to the conclusion that it is optimal for $A1$ to *follow $P2$'s signal* playing $\bar{a}$ for every $A2$'s pure action strategy $\hat{\eta}^2$.

3.) $A1$ receives the signal $\underline{a}$ from both principals. Given these signals and since $q^1_2(y_{22},\bar{a}|\underline{a})=1$, her expected payoff when choosing $a^1\in\{\bar{a},\underline{a}\}$ for a given pure action strategy $\hat{\eta}^2$ of her opponent, is

$$\begin{aligned}&\int_{\alpha_1}q^1_1(\alpha_1,\bar{a}|\underline{a}))[\pi^{\alpha_1}_{a^1\hat{\eta}^2(\bar{a})}\,u^1(y_{11},y_{22},a^1,\hat{\eta}^2(\bar{a}))+(1-\pi^{\alpha_1}_{a^1\hat{\eta}^2(\bar{a})})\,u^1(y_{12},y_{22},a^1,\hat{\eta}^2(\bar{a}))]d\alpha_1\\&+\int_{\alpha_1}q^1_1(\alpha_1,\underline{a}|\underline{a})[\pi^{\alpha_1}_{a^1\hat{\eta}^2(\underline{a})}\,u^1(y_{11},y_{22},a^1,\hat{\eta}^2(\underline{a}))+(1-\pi^{\alpha_1}_{a^1\hat{\eta}^2(\underline{a})})\,u^1(y_{12},y_{22},a^1,\hat{\eta}^2(\underline{a}))]d\alpha_1,\qquad(5)\end{aligned}$$

in which, we again abuse notation and let $\hat{\eta}^2(s)\in\{\bar{a},\underline{a}\}$ be the action that the strategy $\hat{\eta}_2$ prescribes to $A2$ when receiving the signal $s\in\{\bar{a},\underline{a}\}$ from $P1$. To determine $A1$'s optimal actions given her beliefs on $A2$'s behavior, we consider again the relevant four sub-cases.

3a.) $\hat{\eta}^2(\bar{a})=\hat{\eta}^2(\underline{a})=\bar{a}$. In this case, $A1$ gets $\int_{\alpha_1}\left(\frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}}-1\right)\left[q^1_1(\alpha_1,\bar{a}|\underline{a})+q^1_1(\alpha_1,\underline{a}|\underline{a})\right]d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1} 5\pi^{\alpha_1}_{\underline{a}\bar{a}} \left[q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{\bar{a}\bar{a}}$ and $\pi^{\alpha_1}_{\underline{a}\bar{a}}$ are smaller than one for each α_1,

$$\int_{\alpha_1} \left(\frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}} - 1\right) \left[q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1 < \int_{\alpha_1} 5\pi^{\alpha_1}_{\underline{a}\bar{a}} \left[q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1,$$

hence, $A1$ strictly prefers $\underline{a}$ to $\bar{a}$ for every $q^1_1(\alpha_1, \bar{a}|\underline{a}), q^1_1(\alpha_1, \underline{a}|\underline{a})$ and $(\pi^{\alpha_1}_{\bar{a}\bar{a}}, \pi^{\alpha_1}_{\underline{a}\bar{a}})$.

3b.) $\hat{\eta}^2(\bar{a}) = \bar{a}$, $\hat{\eta}^2(\underline{a}) = \underline{a}$. In this case, $A1$ gets $-\int_{\alpha_1} \left[\left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}}\right) q^1_1(\alpha_1, \bar{a}|\underline{a}) + \left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}}\right) q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1} \left[5\pi^{\alpha_1}_{\underline{a}\bar{a}} q^1_1(\alpha_1, \bar{a}|\underline{a}) + \pi^{\alpha_1}_{\underline{a}\underline{a}} q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{a^1a^2} \in [0, 1]$ for every $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ and for every α_1,

$$-\int_{\alpha_1} \left[\left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}}\right) q^1_1(\alpha_1, \bar{a}|\underline{a}) + \left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}}\right) q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1 < \int_{\alpha_1} \left[5\pi^{\alpha_1}_{\underline{a}\bar{a}} q^1_1(\alpha_1, \bar{a}|\underline{a}) + \pi^{\alpha_1}_{\underline{a}\underline{a}} q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1,$$

and $A1$ strictly prefers $\underline{a}$ to $\bar{a}$ for every $q^1_1(\alpha_1, \bar{a}|\underline{a}), q^1_1(\alpha_1, \underline{a}|\underline{a})$ and $(\pi^{\alpha_1}_{\bar{a}\bar{a}}, \pi^{\alpha_1}_{\underline{a}\bar{a}}, \pi^{\alpha_1}_{\bar{a}\underline{a}}, \pi^{\alpha_1}_{\underline{a}\underline{a}})$.

3c.) $\hat{\eta}^2(\bar{a}) = \underline{a}$, $\hat{\eta}^2(\underline{a}) = \bar{a}$. In this case, $A1$ gets $-\int_{\alpha_1} \left[\left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}}\right) q^1_1(\alpha_1, \bar{a}|\underline{a}) + \left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}}\right) q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1} \left[\pi^{\alpha_1}_{\underline{a}\underline{a}} q^1_1(\alpha_1, \bar{a}|\underline{a}) + 5\pi^{\alpha_1}_{\underline{a}\bar{a}} q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1$ by playing $\underline{a}$. Since $\pi^{\alpha_1}_{a^1a^2} \in [0, 1]$ for every $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ and for every α_1, one has

$$-\int_{\alpha_1} \left[\left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}}\right) q^1_1(\alpha_1, \bar{a}|\underline{a}) + \left(1 - \frac{9}{10}\pi^{\alpha_1}_{\bar{a}\bar{a}}\right) q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1 < \int_{\alpha_1} \left[\pi^{\alpha_1}_{\underline{a}\underline{a}} q^1_1(\alpha_1, \bar{a}|\underline{a}) + 5\pi^{\alpha_1}_{\underline{a}\bar{a}} q^1_1(\alpha_1, \underline{a}|\underline{a})\right] d\alpha_1,$$

which leads to the same conclusion of 3b.).

3d.) $\hat{\eta}^2(\bar{a}) = \hat{\eta}^2(\underline{a}) = \underline{a}$. In this case, $A1$ gets $\int_{\alpha_1} \left[\frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}} - 1\right] \left(q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right) d\alpha_1$ by playing $\bar{a}$, and she would get $\int_{\alpha_1} \pi^{\alpha_1}_{\underline{a}\underline{a}} \left(q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right) d\alpha_1$ by playing $\underline{a}$. Since $\left[\frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}} - 1\right] < 0$ for every $\pi^{\alpha_1}_{\bar{a}\underline{a}}$ and α_1, one has

$$\int_{\alpha_1} \left[\frac{9}{10}\pi^{\alpha_1}_{\bar{a}\underline{a}} - 1\right] \left(q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right) d\alpha_1 < \int_{\alpha_1} \pi^{\alpha_1}_{\underline{a}\underline{a}} \left(q^1_1(\alpha_1, \bar{a}|\underline{a}) + q^1_1(\alpha_1, \underline{a}|\underline{a})\right) d\alpha_1,$$

and $A1$ strictly prefers $\underline{a}$ to $\bar{a}$ for every $q^1_1(\alpha_1, \bar{a}|\underline{a}), q^1_1(\alpha_1, \underline{a}|\underline{a})$ and $(\pi^{\alpha_1}_{\bar{a}\underline{a}}, \pi^{\alpha_1}_{\underline{a}\underline{a}})$.

To resume, upon getting $(\underline{a}, \underline{a})$ from both principals, it is optimal for $A1$ to play $\underline{a}$ for every pure action strategy $\hat{\eta}^2$.

4.) $A1$ receives the signal $\bar{a}$ from $P1$ and the signal $\underline{a}$ from $P2$. In this case, her expected payoff can be derived from (5) by substituting $(q^1_1(\alpha_1, \underline{a}|\underline{a}), q^1_1(\alpha_1, \bar{a}|\underline{a}))$ with $(q^1_1(\alpha_1, \underline{a}|\bar{a}), q^1_1(\alpha_1, \bar{a}|\bar{a}))$. As a consequence, to determine $A1$'s optimal actions one can follow the analysis developed in 3a.)-3d.), which leads to the conclusion that it is optimal for $A1$ to follow $P2$'s signal playing $\underline{a}$ for every $A2$'s pure action strategy $\hat{\eta}^2$.

Thus, given $(\hat{\gamma}_1, \hat{\gamma}_2)$, $A1$ has a strictly dominant strategy in playing according to the signal she gets from $P2$ in every action game induced by a $q_1 \in \Delta(\mathcal{Y}_1 \times A^1 \times A^2)$.

We now turn to $A2$'s behavior. Recall that since $A2$ only receives the signal $\bar{a}$ from $P2$ with positive probability, therefore, she effectively forms posterior probabilities only relative to $P1$'s decisions and signals.

Consider first the case in which $A2$ receives the signal $\bar{a}$ from $P1$ and $P2$. Given the equilibrium behaviour of $A1$ in the action game, $A2$ (strictly) prefers to play $\bar{a}$ rather than $\underline{a}$, whenever

$$\int_{\alpha_1} \left[5k + (1-k)5\pi^{\alpha_1}_{\underline{a}\bar{a}}\right] (q_1^2(\alpha_1, \bar{a}|\bar{a}) + q_1^2(\alpha_1, \underline{a}|\bar{a}))d\alpha_1 > \int_{\alpha_1} [8k - 10(1-k)] (q_1^2(\alpha_1, \bar{a}|\bar{a}) + q_1^2(\alpha_1, \underline{a}|\bar{a}))d\alpha_1,$$

that is, whenever

$$\begin{aligned} &\int_{\alpha_1} \left[5k + (1-k)5\pi^{\alpha_1}_{\underline{a}\bar{a}} + 10 - 18k\right] (q_1^2(\alpha_1, \bar{a}|\bar{a}) + q_1^2(\alpha_1, \underline{a}|\bar{a}))d\alpha_1 = \\ &= \int_{\alpha_1} \left[(1-k)5\pi^{\alpha_1}_{\underline{a}\bar{a}} + 10 - 13k\right] (q_1^2(\alpha_1, \bar{a}|\bar{a}) + q_1^2(\alpha_1, \underline{a}|\bar{a}))d\alpha_1 > 0, \end{aligned} \tag{6}$$

which holds for every $\pi^{\alpha_1}_{\underline{a}\bar{a}} \in [0,1]$ if $k \in (0, 10/13)$. Consider next the case in which $A2$ receives the signal $\underline{a}$ from $P1$ and $\bar{a}$ from $P2$. Then, we can rewrite the inequality in (6) by substituting $(q_1^2(\alpha_1, \bar{a}|\bar{a}) + q_1^2(\alpha_1, \underline{a}|\bar{a}))$ with $(q_1^2(\alpha_1, \bar{a}|\underline{a}) + q_1^2(\alpha_1, \underline{a}|\underline{a}))$, and reestablish that, if $k \in (0, 10/13)$, $A2$ strictly prefers $\bar{a}$ to $\underline{a}$.

Hence, given $(\hat{\gamma}_1, \hat{\gamma}_2)$ and $k \in (0, 10/13)$, and for every $q_1 \in \Delta(\mathcal{Y}_1 \times A^1 \times A^2)$, the agents' action game has a unique equilibrium in which both agents play according to the signal they get from $P2$, regardless of the signal received from $P1$.

The corresponding expected payoff to $P2$ is $95k - (1-k)$ which is strictly greater than 5 for every $k > 1/16$. Therefore, setting $k \in (1/16, 10/13)$ in $\hat{\gamma}_2$ as specified i.)-ii.) yields the result.

Step 2. We now consider the case in which $P1$'s probability distribution over his decisions and the signals he sends to agents has an arbitrary support in $(\mathcal{Y}_1 \times S^1 \times S^2)$. That is, $q_1 \in \Delta(\mathcal{Y}_1 \times S^1 \times S^2)$. As a consequence, in the corresponding action game, each agent receives more private signals from $P1$. This however does not alter the agents' equilibrium behaviors, as we show in the next paragraphs.

Let $s_1^i \in S_1^i$ be a signal privately sent by $P1$ to agent $i = 1, 2$ and s_1^{-i} be any array of signals sent by $P1$ to i's opponent. Then, let $\hat{\eta}^2(s_1^2)$ represent the action that the pure strategy $\hat{\eta}_2$ prescribes to $A2$ when receiving the signal $s_1^2 \in S_1^2$ from $P1$, and $q_1^1(\alpha_1, s_1^2|s_1^1)$ be the conditional (joint) probability formed by $A1$ on $P1$'s incentive scheme α_1 and signal s_1^2 to $A2$, having received s_1^1.

In Step 1, we established the result when S_1^i is a binary set of signals for every $i = 1, 2$. We now show that the analysis straightforwardly extends to arbitrary sets S_1^i. Consider first $A1$: we show that for every profile of signals received from principals, she strictly prefers to *play according to P2's signal* for every action strategy of her opponent. Indeed, for each pure action strategy $\hat{\eta}^2$ of $A2$, it is possible to partition the set of $P1$'s signals to $A2$ in two sub-sets: one including all signals that induce $A2$ to play $\bar{a}$, the other those inducing to play $\underline{a}$. Let $\bar{S}_1^2 = \{s_1^2 \in S_1^2 : \hat{\eta}^2(s_1^2) = \bar{a}\}$ and $\underline{S}_1^2 = \{s_1^2 \in S_1^2 : \hat{\eta}^2(s_1^2) = \underline{a}\}$ be such sub-sets. From the view point of $A1$, given $\hat{\eta}^2$, everything happens as if $P1$'s set of signals to $A2$ was binary, with the probability of each of these two signals equal to the sum of the posteriors probabilities of all signals in S_1^i inducing a given action, i.e. $q_1^1(\alpha_1, \bar{a}|s_1^1) = \sum_{s_1^2 \in \bar{S}_1^2} q_1^1(\alpha_1, s_1^2|s_1^1)$ and $q_1^1(\alpha_1, \underline{a}|s_1^1) = \sum_{s_1^2 \in \underline{S}_1^2} q_1^1(\alpha_1, s_1^2|s_1^1)$.

Thus, the optimal behavior of $A1$ can be characterized by extending the analysis of Step 1 to this more general scenario. Consider, as an example, the case in which $A1$ receives the signal $\underline{a}$ from $P2$ and s_1^1 from $P1$: given $\hat{\eta}^2$, her expected payoff by playing $\bar{a}$ will be $-\int_{\alpha_1} \sum_{s_1^2 \in \bar{S}_1^2} q_1^1(\alpha_1, s_1^2|s_1^1) \left(1 - \frac{9}{10}\pi_{\bar{a}\bar{a}}^{\alpha_1}\right) d\alpha_1 - \int_{\alpha_1} \sum_{s_1^2 \in \underline{S}_1^2} q_1^1(\alpha_1, s_1^2|s_1^1) \left(1 - \frac{9}{10}\pi_{\bar{a}\underline{a}}^{\alpha_1}\right) d\alpha_1$, while by playing $\underline{a}$ it will be $\int_{\alpha_1} \sum_{s_1^2 \in \bar{S}_1^2} 5\pi_{\underline{a}\bar{a}}^{\alpha_1} q_1^1(\alpha_1, s_1^2|s_1^1) d\alpha_1 + \int_{\alpha_1} \sum_{s_1^2 \in \underline{S}_1^2} \pi_{\underline{a}\underline{a}}^{\alpha_1} q_1^1(\alpha_1, s_1^2|s_1^1) d\alpha_1$. Since $\pi_{a^1a^2}^{\alpha_1} \in [0, 1]$ for every $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ and for every α_1,

$$-\int_{\alpha_1} \left[\sum_{s_1^2 \in \bar{S}_1^2} \left(1 - \frac{9}{10}\pi_{\bar{a}\bar{a}}^{\alpha_1}\right) q_1^1(\alpha_1, s_1^2|s_1^1) - \sum_{s_1^2 \in \underline{S}_1^2} \left(1 - \frac{9}{10}\pi_{\bar{a}\underline{a}}^{\alpha_1}\right) q_1^1(\alpha_1, s_1^2|s_1^1) \right] d\alpha_1 <$$

$$< \int_{\alpha_1} \left[\sum_{s_1^2 \in \bar{S}_1^2} 5\pi_{\underline{a}\bar{a}}^{\alpha_1} q_1^1(\alpha_1, s_1^2|s_1^1) + \sum_{s_1^2 \in \underline{S}_1^2} \pi_{\underline{a}\underline{a}}^{\alpha_1} q_1^1(\alpha_1, s_1^2|s_1^1) \right] d\alpha_1 \tag{7}$$

and $A1$ strictly prefers $\underline{a}$ to $\bar{a}$ for every $q_1^1(\alpha_1, s_1^2|s_1^1)$ and $(\pi_{\bar{a}\bar{a}}^{\alpha_1}, \pi_{\underline{a}\bar{a}}^{\alpha_1}, \pi_{\bar{a}\underline{a}}^{\alpha_1}, \pi_{\underline{a}\underline{a}}^{\alpha_1})$. The inequality (7) holds for every $\hat{\eta}^2$ and its corresponding $\bar{S}_1^2$ and $\underline{S}_1^2$ sets. The same reasoning applies to the case in which $A1$ receives $\bar{a}$ from $P2$ and some s_1^1 from $P1$.

It remains to show that given such equilibrium behavior of $A1$, $A2$ (strictly) prefers to play $\bar{a}$ rather than $\underline{a}$ regardless of the private signals received from $P1$. Let $s_1^2 \in S_1^2$ be the private signal she receives from $P1$ and $s_1^1 \in S_1^1$ any array of signals that $P1$ sends to her opponent, and recall that she receives $\bar{a}$ from $P2$. Given the equilibrium behavior of $A1$, she (strictly) prefers to play $\bar{a}$ rather than $\underline{a}$, whenever

$$\int_{\alpha_1} \sum_{s_1^1 \in S_1^1} q_1^2(\alpha_1, s_1^1|s_1^2) \left[5k + (1-k)5\pi_{\underline{a}\bar{a}}^{\alpha_1}\right] d\alpha_1 > \int_{\alpha_1} \sum_{s_1^1 \in S_1^1} q_1^2(\alpha_1, s_1^1|s_1^2) \left[8k - 10(1-k)\right] d\alpha_1.$$

The expected payoff to $A2$ is affected by $P1$'s signals only through changes in the conditional probability $\sum_{s_1^1 \in S_1^1} q_1^2(\alpha_1, s_1^1|s_1^2)$. This allows to extend the argument developed in Step 1 to this general case. Thus, given $\hat{\gamma}_1$, any mechanism $\hat{\gamma}_2$ with $k \in (1/16, 10/13)$ yields $P2$ a payoff strictly above 5. ■

The proof establishes that $P2$ achieves a payoff strictly above 5 in any equilibrium of a game with signals G^{MS}. To illustrate its logic, it is useful to first consider the degenerate case in which $\hat{\gamma}_1$ puts positive probability only on one signal. That is, $P1$ does not privately communicate with agents. Then, by posting $\hat{\gamma}_2$, $P2$ induces some incomplete information in the agents' action game. Given their private signals, $A1$ and $A2$ have different posterior probability distributions over the decisions implemented by $\hat{\gamma}_2$. In particular, $P2$ correlates his decisions with the signals in such a way that the signal received by $A1$ gives her perfect information, while the one received by $A2$ is uninformative. The proof points out that the unique Nash equilibrium of the corresponding agents' action game induces a stochastic allocation, i.e. a distribution over A, Y_1 and Y_2, which is not incentive feasible in the absence of private signals. Thus, $\hat{\gamma}_2$ yields $P2$ a payoff greater than 5 even if $P1$ delegates to the agents the choice of his incentive scheme, in such a way that they can tailor the punishment to any $P2$'s choice.[18]In other words, mechanisms based on deviation-reporting messages are not effective to prevent $P2$ from profitably exploiting private communication.

What if $P1$ can additionally send private signals to agents? By doing so, he could generate novel continuation equilibria that harm his opponent, exploiting the correlation between his decisions and the agents' actions. In the example, $A1$'s preferences over actions, for each decision of $P2$, do not depend on $P1$'s decisions neither on $A2$'s choice. The construction of $\hat{\gamma}_2$ guarantees that this feature can be exploited in such a way to induce $A1$ to follow $P2$'s signal no matter the signal she receives from $P1$. Given $\hat{\gamma}_2$ and the induced equilibrium behavior of $A1$, the proof of Proposition 2 shows that $P1$'s signals do not affect $A2$'s equilibrium actions either. The result does not depend on the size of the signals' spaces of the game G^{MS}.[19] Indeed, the proof shows that the reasoning developed for the case in which $P1$ uses a simple binary set of signals extends to the case of an arbitrary number of signals. In addition, the result neither depends on the size of the message set that each agent uses to communicate with principals. In particular, it holds for any M_j^i that is *large*

[18]Observe that a payoff greater than 5 does not belong to the convex hull of $P2$'s payoffs associated to the incentive feasible allocations of games without signals. Hence, it cannot be generated by adding a public correlation device to the competing mechanism game analysed in Section 3.1, as done for instance by Peters and Troncoso-Valverde (2013).

[19]To simplify exposition, the proof of Proposition 2 is developed for the case of finite signal spaces. Furthermore, the assumption that $A^i \subseteq S_j^i$ for every (i, j) is made to guarantee that all principals may send meaningful signals.

in the sense of Yamashita (2010), that is, it includes all direct mechanisms with signals available to principals.

Furthermore, given $(\hat{\gamma}_1, \hat{\gamma}_2)$, the agents' action game exhibits a unique equilibrium for every message they may send to and signal they may receive from $P1$, which guarantees that the proof does not rely on any equilibrium selection argument. That is, there is no "babbling" equilibrium in which agents ignore $P2$'s signal.

Thus, none of the allocations characterized in Proposition 1 can be sustained at equilibrium in a competing mechanism game with signals. A straightforward implication is that equilibria sustained by mechanisms without signals, such as recommendation mechanisms, fail to be robust. This leads to the following:

Corollary 1 *None of the equilibria characterized in Proposition 1, in which principals post the recommendation mechanisms* (γ_1^R, γ_2^R)*, is robust to unilateral deviations of* $P2$ *towards mechanisms with signals.*

To summarize, any equilibrium allocation of a game G^{MS}, in which signals are non-degenerate for at least one principal, is not an equilibrium allocation of the corresponding game G^M, for every collection of message sets M. We next show that principals' private communication plays a role at equilibrium.

3.3 Principals' private communication: equilibrium existence

The following proposition establishes, in the context of the example, equilibrium existence for games with private signals.

Proposition 3 *Consider any game* G^{MS} *in which* M_j^i *is arbitrary and* $A^i \subseteq S_j^i$ *for every* i *and* j*. The payoffs profile* $(2, 79, \frac{11}{3}, 5, 1)$ *can be supported in an equilibrium of* G^{MS} *in which principals play pure strategies.*

Proof. Let $P1$ commit to a degenerate mechanism with signals, $\hat{\gamma}_1$, such that for every array of agents' messages m_1, he plays $\{y_{11}\}$ for every $(a^1, a^2) \in A^1 \times A^2$ and sends to each agent the same signal $\{s\}$ with probability one. Given $\hat{\gamma}_1$, the payoffs to $P2$, $A1$ and $A2$ are reported in Table 5.

Since $\hat{\gamma}_1$ implements a fixed decision irrespective of messages, signals and agents' actions, from the viewpoint of $P2$ finding his best response amounts to solve a single-principal mechanism design problem as in Myerson (1982). Hence, an optimal mechanism can be characterized in terms of a direct mechanism with signals $\hat{\hat{\gamma}}_2$, in which $P2$ commits to the same joint probability distribution

		y_{21}			y_{22}	
		$\bar{a}$	$\underline{a}$		$\bar{a}$	$\underline{a}$
y_{11}	$\bar{a}$	$(95, 10, 5)$	$(\zeta, 3/2, 8)$	$\bar{a}$	$(\zeta, -1/10, 0)$	$(\zeta, -1/10, 8)$
	$\underline{a}$	$(-1, 0, 0)$	$(\zeta, 0, 10)$	$\underline{a}$	$(-1, 5, 5)$	$(\zeta, 1, -10)$

Table 5: The payoff matrix given $\hat{\gamma}_1$

on incentive schemes and actions signalled to agents for every profile of received messages. That is, $\hat{\hat{\gamma}}_2 \in \Delta(\mathcal{Y}_2 \times A)$. As in the single-principal setting of Myerson (1982), direct mechanisms with signals are sufficiently rich to incorporate any randomness in the incentive schemes of $P2$. Hence, when characterizing an optimal mechanism for $P2$ one can safely restrict to joint probability distributions over *deterministic* incentive schemes and signals for $P2$. In addition, in the action game induced by $\hat{\hat{\gamma}}_2$ and by the degenerate mechanism $\hat{\gamma}_1$, it is with no loss of generality to focus on equilibria, in which each agent follows the signal she privately receives from $P2$.

Since $P2$ incurs a loss ζ whenever $A2$ chooses $\underline{a}$, any optimal mechanism for him must put probability zero on signaling the action $\underline{a}$ to $A2$. When designing $\hat{\hat{\gamma}}_2$, $P2$ can exploit the flexibility of an incentive scheme to alleviate the incentive constraints faced by each of the agents. Indeed, the support of his mechanism consists of all the possible combinations of the two signal arrays $(\bar{a}, \bar{a})$ and $(\underline{a}, \bar{a})$ with all deterministic incentive schemes. To simplify notation, let us denote $q_2(\alpha, \bar{a}, \bar{a}) \equiv \bar{k}(\alpha)$ and $q_2(\alpha, \underline{a}, \bar{a}) \equiv \underline{k}(\alpha)$ the joint probabilities attributed by $\hat{\hat{\gamma}}_2$ to the incentive scheme $\alpha \in \mathcal{Y}_2^D$, with $\mathcal{Y}_2^D \subset \mathcal{Y}_2$ being the set of deterministic incentive schemes, and to any of the two relevant profiles of signals.

We next consider the agents' incentive constraints. As for $A1$, when she gets the signal $\bar{a}$ from $P2$, the expected payoff from taking the action $\bar{a}$ has to be no lower than the payoff from taking $\underline{a}$, given the belief on $A2$'s obedience to $P2$. The inequality should be satisfied for each α implemented by $\hat{\hat{\gamma}}_2$ with positive probability when $\bar{a}$ is sent to $A1$. This in turn generates a set of incentive constraints for $A1$. We now show that it is optimal for $P2$ to assign a positive probability $\bar{k}(\alpha)$ only to those incentive schemes α that implement the decision y_{21} for every action chosen by $A1$ when $A2$ chooses $\bar{a}$. That is, to any α such that $\alpha(\bar{a}, \bar{a}) = \alpha(\underline{a}, \bar{a}) = y_{21}$. The corresponding incentive constraint for $A1$ is

$$\frac{\bar{k}(\alpha)}{\sum\limits_{\alpha'' \in \mathcal{Y}_2^D} \bar{k}(\alpha'')} 10 \geq 0, \tag{8}$$

in which $\sum\limits_{\alpha'' \in \mathcal{Y}_2^D} \bar{k}(\alpha'')$ denotes the marginal probability of receiving the signal $\bar{a}$ for $A1$ and zero is the payoff corresponding to the choice $\underline{a}$. Indeed, any incentive scheme α such that either $\alpha(\bar{a},\bar{a}) \neq \alpha(\underline{a},\bar{a})$ or $\alpha(\bar{a},\bar{a}) = \alpha(\underline{a},\bar{a}) = y_{22}$ induces an incentive constraint for $A1$ when she receives $\bar{a}$ that is different from (8). Yet, one can check that every $\bar{k}(\alpha)$ satisfying any of those constraints also satisfies (8), but the converse may not be true. This implies that, when designing $\hat{\hat{\gamma}}_2$, $P2$ finds optimal to set $\bar{k}(\alpha) > 0$ only for those α such that $\alpha(\bar{a},\bar{a}) = \alpha(\underline{a},\bar{a}) = y_{21}$. By doing so, $P2$ effectively neutralizes the incentive constraints of $A1$ when she receives the signal $\bar{a}$.

The set of incentive constraints for $A1$ when she receives $\underline{a}$ from $P2$ can be analyzed in the same way. Specifically, we show that, in this case, it is optimal to put a positive probability $\underline{k}(\alpha')$ only on those α' such that $\alpha'(\bar{a},\bar{a}) = \alpha'(\underline{a},\bar{a}) = y_{22}$. Indeed, the corresponding incentive constraint for $A1$ would be

$$\frac{\underline{k}(\alpha')}{\sum_{\alpha''\in\mathcal{Y}_2^D}\underline{k}(\alpha'')}5 \geq \frac{\underline{k}(\alpha')}{\sum_{\alpha''\in\mathcal{Y}_2^D}\underline{k}(\alpha'')}(-\frac{1}{10}). \tag{9}$$

Once again, we remark that any incentive scheme α' such that either $\alpha'(\bar{a},\bar{a}) \neq \alpha'(\underline{a},\bar{a})$ or $\alpha'(\bar{a},\bar{a}) = \alpha'(\underline{a},\bar{a}) = y_{21}$ induces an incentive constraint for $A1$ when she receives $\underline{a}$ that is different from (9). Yet, one can check that every $\underline{k}(\alpha')$ satisfying any of those constraints also satisfies (9), but the converse may not be true. This implies that, when designing $\hat{\hat{\gamma}}_2$, $P2$ finds optimal to set $\underline{k}(\alpha') > 0$ for those α' such that $\alpha'(\bar{a},\bar{a}) = \alpha'(\underline{a},\bar{a}) = y_{22}$ therefore neutralizing the incentive constraints of $A1$ when she receives the signal $\underline{a}$.
An optimal mechanism for $P2$ hence consists of a distribution $(\bar{k}(\alpha),\underline{k}(\alpha'))$ which assigns probability $\bar{k}(\alpha)$ to any α such that $\alpha(\bar{a},\bar{a}) = \alpha(\underline{a},\bar{a}) = y_{21}$ together with signals $(\bar{a},\bar{a})$ and probability $\underline{k}(\alpha')$ to any α' such that $\alpha'(\bar{a},\bar{a}) = \alpha'(\underline{a},\bar{a}) = y_{22}$ together with signals $(\underline{a},\bar{a})$.

Let us now consider the incentive constraints of $A2$. Since she only gets the signal $\bar{a}$ from $P2$, she cannot update her prior probabilities. Thus, given $\hat{\hat{\gamma}}_2$, her decisions depend on $\bar{k}(\alpha)$ and $\underline{k}(\alpha')$. We now show that it is optimal for $P2$ to set $\alpha(\bar{a},\underline{a}) = y_{21}$ and $\alpha'(\underline{a},\underline{a}) = y_{22}$. In this case, an incentive constraint for $A2$ can be written as

$$\frac{\bar{k}(\alpha)}{K(\alpha'')}5 + \frac{\underline{k}(\alpha')}{K(\alpha'')}5 \geq \frac{\bar{k}(\alpha)}{K(\alpha'')}8 + \frac{\underline{k}(\alpha')}{K(\alpha'')}(-10) \tag{10}$$

in which $K(\alpha'') \equiv \sum_{\alpha''\in\mathcal{Y}_2^D}\bar{k}(\alpha'') + \underline{k}(\alpha'') = 1$ denotes the marginal probability of receiving the signal $\bar{a}$ for $A2$. Observe that the left-hand side of (10) is fully determined by the conditions on $\alpha(\bar{a},\bar{a}) = y_{21}$ and $\alpha'(\bar{a},\bar{a}) = y_{22}$ specified above. In addition, one can check that the expression on the right-hand side is only affected by $\alpha'(\underline{a},\underline{a})$, and it is minimized when $\alpha'(\underline{a},\underline{a}) = y_{22}$.

We complete the description of an optimal mechanism $\hat{\hat{\gamma}}_2$ specifying the decision that α associates to the agents' actions $(\underline{a}, \underline{a})$, and the decision that α' associates to the actions $(\bar{a}, \underline{a})$. With no loss of generality, we set $\alpha(\underline{a}, \underline{a}) = y_{21}$ and $\alpha'(\bar{a}, \underline{a}) = y_{22}$. Indeed, neither $A1$ nor $A2$'s incentive constraints are affected by these decisions and $P2$'s payoff is constant and equal to ζ over his decisions when $A2$ chooses $\underline{a}$.

Thus, when $P1$ posts the degenerate mechanism with signals $\hat{\gamma}_1$, it is optimal for $P2$ to post $\hat{\hat{\gamma}}_2$ which involves a correlation between signals and (uncontingent) incentive schemes. The corresponding correlated distribution implemented by $\hat{\hat{\gamma}}_2$ reduces to the two joint probabilities $q_2(\alpha, \bar{a}, \bar{a}) = \bar{k}$ and $q_2(\alpha', \underline{a}, \bar{a}) = \underline{k}$, with $\alpha(a^1, a^2) = y_{21}$, $\alpha'(a^1, a^2) = y_{22}$ for all $(a^1, a^2) \in \{\bar{a}, \underline{a}\}^2$ and $\underline{k} = 1 - \bar{k}$. Therefore, the constraints in (8), (9) and (10) become:

$$\begin{aligned} 10\bar{k} &\geq 0 && \text{which holds for every } \bar{k} \geq 0 \\ 5\underline{k} &\geq -\tfrac{1}{10}\underline{k} && \text{which holds for every } \underline{k} \geq 0 \\ 5(\bar{k} + \underline{k}) &\geq 8\bar{k} - 10\underline{k}. && \end{aligned} \tag{11}$$

An optimal mechanism with signals for $P2$ should maximize his expected payoff $V_2 = 95\bar{k} - \underline{k}$ subject to (11). The unique solution involves $\bar{k} = \frac{15}{18}$ and $\underline{k} = \frac{3}{18}$, yielding $P2$ a payoff of

$$95\bar{k} - \underline{k} = 79 > 5. \tag{12}$$

The corresponding equilibrium payoffs for all players are $(2, 79, \frac{11}{3}, 5, 1)$ as claimed. ■

The result shows the existence of equilibrium payoffs that do not belong to the set characterized in Proposition 1. The proof of Proposition 3 crucially exploits the fact that $P1$'s equilibrium strategy consists of a degenerate mechanism. This in turn allows to restrict attention to direct mechanisms with signals for $P2$. That is, given $P1$'s strategy, for every set of agents' messages and principals' signals, any allocation which is optimal from the viewpoint of $P2$ can be supported by letting $P2$ privately recommend an action to each agent, and requiring agents to obey such recommendations. Characterizing an optimal mechanism in this class is quite involved since one has to consider the set of joint probability distributions over *incentive schemes* and signals sent to $A1$ and $A2$.

One should observe that the mechanism $\hat{\hat{\gamma}}_2$, which is optimal *given* that $P1$ plays $\hat{\gamma}_1$, turns out to be formally equivalent to the direct mechanism with signals for $P2$ exhibited in the proof of Proposition 2. This allows to directly relate the result of Proposition 3 with that of Proposition 2.

The proof of Proposition 3 shows that, if $P1$ posts the degenerate mechanism $\{y_{11}\}$, and $A1$ plays in accordance to the signal received from $P2$, then any $\bar{k} \leq \frac{15}{18}$ induces $A2$ to play $\bar{a}$. To establish Proposition 2, instead, we have to identify the values of $\bar{k}$ which yield the same implications *for all mechanisms posted by* $P1$. As shown in (6), this requires setting $\bar{k} < \frac{10}{13} < \frac{15}{18}$, which implies that the corresponding payoff to $P2$ is bounded above by $95\frac{10}{13} - \frac{3}{13} = \frac{947}{13} < 79$.

The same reasoning followed in the proof of Proposition 3 can be iterated to determine the optimal (equilibrium) mechanism of $P2$ were $P1$ posting any other deterministic mechanism independent of messages and signals. For instance, if $P1$ commits to the degenerate mechanism $\{y_{12}\}$, it can be shown that an optimal mechanism for $P2$ yields him a payoff of 80.[20] This shows that, in the context of the example, any G^{MS} game exhibits multiple equilibrium allocations.

4 Discussion

1. Our analysis has two main implications. On the one hand, the equilibria of any game G^M are not robust to unilateral deviations of a principal to mechanisms with signals. This suggests that the general construction derived in Epstein and Peters (1999) may fail to reproduce all communication opportunities between principals and agents. On the other hand, none of the equilibrium allocations of a game in which *all* principals can privately communicate can be supported at equilibrium when this private communication is unfeasible. This suggests that the restriction to one-sided private communication is key to establish folk-theorem-like results in the spirit of Yamashita (2010).

2. We consider the simple scenario in which there is no (exogenous) incomplete information and agents take fully observable actions. Introducing observable actions is a convenient way to model agents' participation decisions, as also done by Epstein and Peters (1999).[21] Indeed, our example can be casted in the two-agents framework of Epstein and Peters (1999), in which each agent is restricted to participate with at most one principal and communication is not constrained by participation decisions. To do so, one should interpret the action $\bar{a}$ as participating with $P1$ but not with $P2$, the action $\underline{a}$ symmetrically, and let the strategy of not participating with either principal be dominated.

The possibility for principals to take decisions contingent on agents' actions is not crucial for our result. First, as remarked in Section 3.1, in the absence of principals' private communication any feasible allocation yields $P2$ a payoff *smaller* than 5. This is a fortiori true when agents' actions are

[20] The detailed derivation of this result is available from the authors.
[21] See Epstein and Peters (1999), pp. 123-125.

not observable. In this case, a direct mechanism for principal j is a flat incentive scheme associating the same decision to all actions, which implies that the corresponding set of feasible allocations is included in Z^{IF}. Second, the mechanism with signals $\hat{\gamma}_2$ used in the proof of Proposition 2 allows $P2$ to get a payoff *greater* than 5 without conditioning on agents' actions.

3. In the light of the former observation, one could wonder whether mechanisms with signals keep playing a key role in pure incomplete information settings, with agents taking no actions. This is the situation considered in Yamashita (2010), who postulates that each agent participates with all principals from the outset. To answer this question, observe that, when information is incomplete and principals play recommendation mechanisms, agents take two communication decisions. First, they recommend to each principal the direct mechanism he should post; second, they simultaneously report a type to each principal. From the viewpoint of a given principal j, the messages (types) that agents send to his opponents can be seen as hidden actions. Indeed, by selecting a profile of decisions in each of the direct mechanisms posted by principals $-j$, such messages may *indirectly* affect principal j's payoff. He may therefore gain by generating uncertainty among agents when they play their message game, using the same logic of our example. That is, principal j may design a mechanism with signals to be privately sent to each agent *before* he receives agents' messages (types). The corresponding continuation equilibrium over messages may induce a correlation between principals' decisions that cannot be reproduced without private signals.

4. The example shares with Yamashita (2010) the focus on recommendation mechanisms. An implication of Proposition 2 is that recommendation mechanisms have a limited power in preventing $P2$ from achieving a payoff above 5 at equilibrium if he uses mechanisms with signals. A relevant issue is whether the result extends to equilibria featuring more sophisticated communication from agents to principals, possibly involving more than one stage.[22] In principle, $P1$ could exploit the additional information he may receive from agents to punish $P2$ in a more effective way. Specifically, $P1$ may set up a further round of communication with agents, asking them to communicate the private information generated by the mechanism with signals $\hat{\gamma}_2$, and commit to modify his decision accordingly. This opportunity, however, is not effective in the example since, for any $\hat{\gamma}_1$, the unique continuation equilibrium of the agents' action game induced by $\hat{\gamma}_2$ is not affected by any further change in the joint distribution q_1.[23]

[22] Lemma 2 in Yamashita (2010) guarantees that recommendation mechanisms are sufficiently flexible to reproduce all the punishments against a deviating principal j which can be generated by arbitrary message spaces of his opponents.

[23] We thank Mike Peters for raising this issue to our attention.

5. The G^M game in which each M^i_j space is a singleton plays a central role in economic applications. In this game, which we denote G^D, competition between principals takes place absent any private communication, and principals post direct mechanisms, which are equivalently labelled pay-for-effort contracts. The game G^D provides, in particular, a generalized version of the traditional models of lobbying of Bernheim and Whinston (1986), Dixit et al. (1997) and Prat and Rustichini (2003). It is therefore a relevant question from the viewpoint of applications whether the equilibria of G^D survive when principals deviate to more complex mechanisms involving some communication. Theorem 1 in Han (2007) provides a positive answer, identifying a set of equilibria that are robust against unilateral deviations to mechanisms *with no signals*. These are the pure strategy *strongly robust* equilibria of G^D, that is, the SPNE in which no principal can profitably deviate to a direct mechanism, regardless of the continuation equilibrium selected by agents.[24] Thus, a strongly robust equilibrium of G^D is also an (strongly robust) equilibrium of any G^M game. Going back to the example, recall that there exists an incentive feasible allocation yielding $P2$ his *maximal* payoff of 5 (Remark 2). Then, as an implication of Lemma 2, this allocation can be supported in a strongly robust equilibrium of G^D. At equilibrium, $P1$ plays y_{12} when observing the actions $(\underline{a}, \bar{a}, \bar{a})$, and y_{11} otherwise; $P2$ plays y_{21} when observing the actions $(\underline{a}, \bar{a}, \bar{a})$, and y_{22} otherwise; A1 plays $\underline{a}$, A2 and A3 play $\bar{a}$, respectively. It hence follows by Theorem 1 in Han (2007) that these behaviors constitute an equilibrium in any G^M game. At the same time, however, the proof of Proposition 2 shows that, if $P1$ plays the mechanism above, then $P2$ can profitably deviate to the mechanism with signals $\hat{\gamma}_2$. Thus, posting these direct mechanisms does not constitute an equilibrium in a game with signals G^{MS}. Overall, this suggests that pure strategy equilibria of complete information games in which principals post pay-for-effort contracts may not be robust against unilateral deviations towards arbitrary indirect mechanisms.

6. Our result crucially exploits the presence of several agents. In single-agent environments, following a principal's deviation to a mechanism with signals, any correlation between the agent's actions and his opponents' decisions can be reproduced using mechanisms without signals.

5 Conclusion

This paper shows that principals' private communication is key for equilibrium characterization in competing mechanism games even under complete information. Since principals cannot in general

[24] See Han (2007), p. 613, for a formal definition of strongly robust equilibria. The result of his Theorem 1 does not extend to equilibria in which principals play mixed strategies, as he shows in Example 1.

be prevented from privately communicating with agents, further theoretical work may be needed to identify a *universal* set of mechanisms for principals in these contexts.

As a preliminary step, one may want to identify a *safe* class of mechanisms supporting *robust* equilibria. To be relevant for applications, the corresponding messages and signals must be sufficiently simple and tractable. In this respect, a natural candidate is the class of direct mechanisms introduced in Myerson (1982) for generalized principal-agent problems, which we have here denoted direct mechanisms with signals. Under complete information, a mechanism in this class requires that the set of signals available to each principal coincides with the set of agents' actions. This choice, however, encounters two main obstacles. The first one is immediate to identify: since an agent can receive conflicting signals from different principals, there is no obvious counterpart to the notion of *obedience* to a principal's recommendation. This would make the characterization exercise very complex, since one cannot straightforwardly rely on incentive compatibility constraints when considering a continuation game played by agents. The second one is more fundamental, and concerns the robustness of equilibria supported by such simple signals. To describe it, consider a principal, say j, whose opponents post a direct mechanism with signals. Principal j may find profitable to elicit the agents' private information embedded in all the signals they receive. To do so, he would need to make his private communication contingent on each array of opponents' signals, a construction that requires an enlarged set of signals for him. In these circumstances, identifying a robust equilibrium may be very demanding. Indeed, some $-j$ principal may further find profitable to make his signals contingent on the (contingent) signals of principal j, which potentially leads to an infinite regress problem similar to that described by Epstein and Peters (1999). The above considerations constitute a challenge for future research.

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