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Learning about profitability and dynamic cash management*

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Abstract: We study a dynamic model of a firm whose shareholders learn about its profitability, face costs of external financing and costs of holding cash. The shareholders' problem involves a notoriously challenging singular stochastic control problem with a two-dimensional degenerate diffusion process. We solve it by means of an explicit construction of its value function, and derive a corporate life-cycle with two stages: a "probation stage" where it is never optimal for the firm to issue new shares, and a "mature stage" where the firm resorts to the market whenever needed. The cash target level is non-monotonic in the belief about the profitability and reaches its highest value on the edge between the two stages. It follows new insights on the firm's volatility and its payout ratio which depend on the firm's stage in its life cycle.

JEL classification: C02; C11; C61; G32; G33; G35

Keywords: Corporate cash management; Corporate life cycle; Learning; Singular control

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1 Introduction

In this paper, we explain how learning about profitability impacts corporate cash management. While the trade-off between the costs and benefits of holding cash has received much attention in recent literature, little is known on the dynamics of this trade-off in a setting where profitability is difficult to ascertain. Nevertheless, this issue is key for corporations that are uncertain of the profitability of their project and face important external financing constraints. The dynamic interaction between learning about profitability and cash management is most relevant for young firms developing new technologies, conducting intensive research and development (R&D) and innovative activities.[1] For those firms, information problems and lack of collateral value of intangible assets make external capital very costly. This leads them to finance their activities with internal cash flow and to issue stock only when cash flow is exhausted.[2]

With few exceptions that we comment below, existing studies develop corporate cash models in a complete information setting and typically remain silent on the intertwining between holding cash and learning about profitability. Our paper contributes to close this gap. We develop a stylized continuous-time model of an all-equity firm confronted with three frictions: imperfect information about profitability, external financing costs and costs of holding cash. Cash constrained shareholders observe at any time realized earnings and update theirs beliefs about the firm's profitability. They control the dynamics of cash through issuance and payment policies. In such a framework, shareholders must cope with both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project).

Formally, we solve a new two-dimensional singular control problem where the state variables are the controlled cash reserves and the profitability prospects resulting from Bayesian learning about the actual profitability. The problem is highly nontrivial. Intuitively, a positive shock to earnings increases the profitability prospects. This should facilitate external financing and induce the firm's management to lower cash target levels. Nevertheless, a firm has more to lose from liquidity constraints when profitability prospects are high than it does when they are low. This may induce the firm's management to accumulate more cash when the profitability prospects increase. We spell out these interactions and show that they result in a two-stage corporate life-cycle dynamics of cash holdings that stems from the optimal equity issuance, payout and liquidation policies that we derive analytically. A rich set of implications follows. We highlight our primary findings here.

Issuance occurs when cash reserves are depleted if and only if the cost of issuance is not

[1] A striking textbook case is the funding crisis Intel faced in the early 1980s. At that time, the large potential of microprocessors was difficult to realize, which is a main reason external financing was extremely costly for Intel, as for instance documented in Passov (2003). We refer to Hall and Lerner (2010) and Kerr and Nanda (2015) for surveys on the financing of innovation and the role of learning in running innovative projects.

[2] Brown, Fazzari and Petersen (2009) point out this feature and find evidence that young, high-tech firms almost entirely financed by cash flow or public share issues explain most of the 1994 to 2004 aggregate R&D cycle. More recently, Graham and Leary (2018) find evidence that a large number of new public Nasdaq firms from 1980 to 2000 were holding large amounts of cash. They find that this effect was most pronounced among unprofitable, largely debt-free, high-growth, and high-volatility firms operating in the health care or high-tech industries.

too high and profitability prospects are larger than an endogenous threshold. Above that threshold, we say that the firm is in the "mature stage": shareholders optimally issues new shares whenever needed, precisely when cash reserves are depleted. Below that threshold, we say that the firm is in the "probation stage": shareholders find never optimal to issue new shares and liquidate the firm when cash is exhausted. Because of liquidity shocks, the firm can go back and forth between the two stages. If the cost of issuance is high, the firm never issues new shares and is therefore limited to the probation stage.

The uncertainty about the firm's actual profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its profitability. We establish that a continuous non-monotonic function of the profitability prospects, *the dividend boundary*, characterizes the cash target levels. We show that when cash reserves reach the dividend boundary, shareholders pay out as dividends a fraction of the cash above the dividend boundary and reinvest the complement into the firm. The fraction that shareholders receive corresponds to the payout ratio of the firm and is also a function of the profitability prospects. Our theoretical analysis yields dynamics of cash holdings that are drastically different in the two different regimes of the corporate life cycle.

In the probation stage, the precautionary motive for holding cash is strong because the threat of liquidation. The dividend boundary is increasing in the profitability prospects. The payout ratio is slightly increasing in the profitability prospects and takes very low value, which means that the firm pays little in dividends. We establish that the firm reaches its maximum cash target level on the edge between the probation stage and the mature stage. Shareholders of a firm entering into the mature stage have built a large amount of cash reserves and have increased profitability prospects. They optimally decide to initiate dividend payments. This causes a discontinuity in the payout ratio, the firm dis-saves and uses its reserves to pay more dividends than its last profit. Then, payout ratios and cash target levels decrease as profitability prospects increase and tend to their values of the complete information benchmark of our model. We show that target levels, increase with the cost of external financing and that a high-cost firm dis-saves more aggressively when it enters into the mature stage. Therefore, our learning model provides in a unified setting a theoretical ground for empirical evidences that, dividend policy, seasoned equity offerings, cash flow patterns are related to a firm's life-cycle stage.[3]

Additional economic insights follow from the model analysis. The model shows that the firm's volatility depends on the firm's stage in its life cycle. In the probation stage, the model predicts a positive relationship between the firm's value and its volatility, while the relationship is negative for firms with proven profitability. On the edge between the probation stage and the mature stage, the volatility of the firm is an inverted U-shaped function of the value of the firm. Overall, our model suggests that most changes in the features of a firm's key indicators (volatility, cash target levels, payout ratios) occur at transition phases between life-cycle stages.

All these results are grounded on a new contribution to the literature on stochastic control. We solve a two-dimensional Bayesian adaptative control problem that combines singular control and stopping.[4] In stochastic control theory, the route to obtaining value functions

[3]We relate our results to the empirical literature in section 5.

[4]The literature on two-dimensional explicitly solvable control and stopping problems relies mainly on

and optimal control policies involves two steps: (i) derive a Hamilton-Jacobi-Bellman (HJB) equation whose solution gives a candidate value function, and, as a by-product, a candidate optimal policy, if any, (ii) leaning on regularity properties of the candidate value function, apply a verification lemma based on the Itô's formula which asserts that the candidates are actually the value function and the optimal control. This procedure is anything but trivial. Academics have provided a lot of efforts to get sets of simple conditions that guarantee the regularity of the value function and the existence of optimal strategies for one-dimensional stopping/control problems.[5] Difficulties are increasing in a two-dimensional framework: a closed form solution to the HJB equation cannot generally be provided and, deriving required regularity properties to apply the verification lemma and to prove the existence of an optimal control is very challenging. These technical difficulties reduce the scope of two-dimensional corporate cash models, which, for the largest majority, rely on numerical approximations that unfortunately do not characterize optimal policies. In contrast, we solve our two-dimensional control problem by means of an explicit construction of its value function and prove all the required regularity properties. We fully characterize the optimal issuance and dividend policies which involve the analysis of three related highly non linear equations. A first equation defines the threshold about profitability prospects above which the firm optimally issues new shares. An intuitive condition of convergence towards the value function associated to the complete information benchmark of our model dictates this threshold. Two other equations, one for each stage, characterize the dividend boundary function. In particular, an ordinary differential equation, which terminal condition accounts for the issuance policy, characterizes the dividend boundary in the probation stage. The dividend boundary is continuous, non-monotonic and reaches its maximum at a point where it is not differentiable. This non-differentiability drives the discontinuity of the payout ratio at the edge between the two stages. To the best of our knowledge, these results are unique to our study. They allow us to provide a rigorous model analysis and to obtain the rich set of results that we summarized above.

Relationship to the literature. Jeanblanc and Shiryaev (1995), Radner and Shepp (1996) have set the benchmark case for the analysis of corporate cash management in continuous time. The cumulative net cash flow generated by the firm follows an arithmetic Brownian motion. The constant drift represents the firm's profitability per unit of time, and the Brownian shock is interpreted as a liquidity shock. External financing is costly, which creates a precautionary demand for cash. Agency costs of free cash flow create a cost of carrying cash. This results in a unique optimal payout policy that requires paying shareholders 100% of earnings beyond an endogenous constant cash target level. [6]

These contributions have been extended in a number of directions. Lokka and Zervos (2008) introduce issuance costs into the analysis. Décamps, Mariotti, Rochet and Villeneuve (DMRV) (2011) study the interaction between cash management, agency costs, issuance costs and stock price; Bolton, Chen and Wang (2011) extend the model to the case of flexible firm size in order to study the dynamic patterns of corporate investment. Bolton,

the study of leading examples. See e.g Benes, Shepp and Witsenhausen (1980), Karatzas, Ocone, Wang and Zervos (2000), Peskir and Shiryaev (2006) Chapter 4, Guo and Zervos (2010).

[5]See for instance Strulovici and Szydlowski (2015).

[6]Influential empirical papers driving the theory are Opler, Pinkowitz and Stulz (1999) and Bates, Kahle and Stulz (2009).

Chen and Wang (2013) and Hugonnier, Malamud and Morellec (2014) introduce capital supply uncertainty and the necessary time needed to secure outside funds into the analysis. Décamps, Gryglewicz, Morellec and Villeneuve (2017) assume that the firm's operating cash flow is proportional to profitability, the dynamics of which are governed by a geometric Brownian motion. This leads to a dividend boundary that is linear and increasing in profitability. Babenko and Tserlukevich (2021) study the relation between investment under financing constraints and optimal risk management policy. All these contributions develop one-dimensional models (or can be reduced to one-dimensional models thanks to scaling properties).

Anderson and Caverhill (2012), Murto and Tervio (2014), Bolton, Wang and Yang (2019), Reppen, Rochet and Soner (2020) develop in different settings two-dimensional corporate cash models with random profitability. As in our study, the firm's decision depends upon two state variables, the current profitability and the current level of liquid assets. In contrast to our study, information about the parameters of the model is complete. In Anderson and Caverhill (2012) and Reppen, Rochet and Soner (2020) the long-term profitability of the firm's project corresponds to the mean parameter of the drift of an Ornstein Uhlenbeck process. In Murto and Tervio (2014) and Bolton, Wang and Yang (2019), the long-term profitability corresponds to the drift of a Geometric Brownian motion that models the earnings fundamentals. These models share important common features. First, technical difficulties do not allow them to characterize the dividend boundary that stems from the trade-off between the costs and benefits of holding cash. Its existence remains a guess and the analysis of the model relies on a numerical treatment of the HJB equation that underlies the shareholders' problem. Second, the models generate high cash target levels at the profitability threshold that triggers liquidation. The intuition is simple. The long-term profitability is known and positive. To avoid costly liquidation or issuance, shareholders increase cash target levels when the current profitability decreases. In contrast, in our learning model, the cash target level takes its minimum (which is equal to zero) at the level of profitability prospect below which the project is liquidated. Here also the intuition is simple. Shareholders decrease cash target levels when they become more and more convinced that the firm's project is not profitable. Under the optimal policy, there is no more cash inside the firm when it is liquidated. Furthermore, we provide an analytical characterization of the dividend boundary. This function is single peaked when issuance costs are low and reaches its maximum on the edge between the two stages of the corporate life cycle. It is increasing when the issuance costs are high because of the fear of having to liquidate a profitable project. Therefore, the two approaches correspond to different economic issues and generate different results. We take the view of a firm whose profitability is difficult to ascertain. Shareholders estimate it from cash flow realizations. In our model, cash allows to learn about the actual profitability of the firm's project and to avoid costly liquidation and/or issuances.

Our study is naturally related to the corporate finance literature that emphasizes the role of learning about profitability and its importance for corporate decision-making. Pastor and Veronesi (2003) study stock prices in a model in which shareholders of an all-equity firm learn about profitability over time. Their model avoids both the liquidity and liquidation issues, assuming a peculiar dividend strategy that maintains a positive book value of the firm's equity at any time. DeMarzo and Sannikov (2017) study a dynamic contracting model with learning about the profitability of the firm. In their model, asymmetric information arises endogenously because by shirking, an entrepreneur can distort the beliefs of investors about the project's profitability. The paper studies the relationship between incentives and learning.[7] Our focus is different. We do not model hidden actions. In our model, information is incomplete but symmetric between shareholders. We study the interplay between the evolution of profitability prospects and the evolution of the trade-off between the cost and benefit of holding cash.

Our problem is very simple in its formulation and is a natural extension of pioneering models. In our paper the drift parameter of the cumulative cash flow process is a random variable that can only take two opposite values over which shareholders learn over time. Our study is closely related to Gryglewicz (2011) and De Angelis (2020) who consider a similar setting. In Gryglewicz (2011) there are no frictions inside the firm, so holding cash is not costly. Gryglewicz (2011) studies how this framework impacts the optimal capital structure that results from the trade-off between tax shields and bankruptcy costs when equity financing is either costless or infinitely costly. De Angelis (2020) considers cash-carry costs and infinite issuance costs. De Angelis (2020) establishes a link between optimal dividends with partial information and the so called problems of optimal stopping with creation. This connection allows to establish regularity properties of the value function and to prove for the first time existence and uniqueness of a dividend boundary in a two-dimensional model with cash-carry costs. However, there is no explicit construction of the value function nor analytical characterization of the dividend boundary, which limits the economic analysis. In our paper, holding cash is costly and issuing new shares has a finite cost. We provide an explicit construction of the value function and of the optimal issuance and payment policies, which is unique to our study.

The paper is organized as follows. We lay out the model in Section 2. Section 3 studies benchmarks in which shareholders face profitability and liquidity concerns separately. Section 4 solves the model in a closed form and presents the optimal corporate policies. Section 5 develops the model analysis and the relation to the empirical literature. Section 6 further comments our main assumptions, discusses robustness issues and concludes. All the proofs are in the Appendix.

2 The model

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a one-dimensional Brownian motion $W = \{W_t; t \geq 0\}$. On the same probability space, we also have a random variable Y which is independent of W and takes either of the two values $-\mu < 0 < \mu$.

2.1 Learning

A firm has a single investment project that generates random cash flows over time. The cumulative cash flow process $\{R_t; t \geq 0\}$ follows an arithmetic Brownian motion with unknown profitability Y and known variance σ

$$dR_t = Y\,dt + \sigma\,dW_t, \quad t \geq 0.$$

[7]See also the related studies Prat and Jovanovic (2014) and He, Wei, Yu and Gao (2017).

The firm is held by risk-neutral shareholders who observe the cumulative cash flow process $R = \{R_t; t \geq 0\}$. We denote by $(\mathcal{F}_t^R)$ the filtration generated by the cash flow process to model the flow of information available to shareholders.[8] The conditional expectation

$$Y_t := \mathbb{E}[Y \mid \mathcal{F}_t^R], \tag{1}$$

defines the profitability prospects at time t and $Y_0 \in (-\mu, \mu)$, the initial profitability prospects. It is well-known that the process $\{Y_t; t \geq 0\}$ satisfies the filtering equation[9]

$$dY_t = \frac{1}{\sigma}(\mu^2 - Y_t^2)dB_t, \tag{2}$$

where the so-called innovation process $\{B_t; t \geq 0\}$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t^R; t \geq 0\}$ and defined as

$$dB_t = \frac{1}{\sigma}(dR_t - Y_t dt). \tag{3}$$

The cumulative cash flow process is a sufficient statistic for Bayesian updating. Specifically, a direct application of Itô's formula yields the relation

$$dR_t = d\phi(Y_t), \tag{4}$$

where the function $\phi(y) = \dfrac{\sigma^2}{2\mu} \ln \left(\dfrac{\mu + y}{\mu - y} \right)$ is increasing on $(-\mu, \mu)$. Finally, we obtain from (3) that

$$\mathbb{E}\left[\int_0^\infty e^{-rs}\, dR_s\right] = \mathbb{E}\left[\int_0^\infty e^{-rs} Y_s\, ds\right] \leq \frac{\mu}{r}.$$

Therefore, the present value of the future cash flows under partial information is lower than the present value of future cash flows of a project with observed profitability μ.

2.2 The shareholders' problem

Risk-neutral shareholders discount future payments at the risk-free interest rate $r > 0$ and must keep positive liquid reserves at all times if they want to avoid liquidation. The model builds on the standard *cost versus benefit trade-off of holding cash.* The firm accumulates cash for precautionary motives in a costly external financing environment. We allow the firm to increase its cash holdings or cover operating losses by raising funds in the capital markets. External financing involves a proportional cost $p > 1$: for each dollar of new shares issued, the firm only receives $1/p$ dollars in cash. Because of internal frictions such as taxes and/or agency costs of free cash flow, carrying cash is costly. Shareholders can reduce these costs by deciding to distribute cash. To capture in a simple and tractable way carrying costs, we

[8] Two filtrations on $(\Omega, \mathcal{F}, \mathbb{P})$ are worth emphasizing: one is the filtration generated by Y and W which corresponds to the information available to the modeller. The other is the filtration generated by R. The process W is not $(\mathcal{F}_t^R)$ adapted and thus is not a Brownian motion under this filtration.

[9] See, for instance, Liptser and Shiryaev (2001), Theorem 12.7 for Equations (2) and (3).

consider that the cash inside the firm is not remunerated.[10] In addition to this trade-off, shareholders do not know the profitability, positive or negative, of the firm's project and use realized earnings to learn about the actual profitability. Thus, shareholders face both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project).

Formally, at each date, shareholders decide whether to continue the project, whether to distribute dividends and whether to issue new shares. For simplicity, we assume that the liquidation value of the project is equal to 0. Let $D = \{D_t; t \geq 0\}$ and $I = \{I_t; t \geq 0\}$ be two $(\mathcal{F}_t^R)_{t\geq 0}$-adapted, nondecreasing, right-continuous processes with $D_{0-} = I_{0-} = 0$. The process $D = \{D_t; t \geq 0\}$ represents the cumulative amount of dividends paid by the firm up to time t, while $I = \{I_t; t \geq 0\}$ represents the cumulative amount of equities issued by the firm up to time t. Under the policy (D, I), the cumulative cash reserve process $X = \{X_t; t \geq 0\}$ evolves according to the dynamics

$$dX_t = dR_t + \frac{dI_t}{p} - dD_t. \tag{5}$$

Thus, dX_t, the cash reserves at time t, corresponds to the operating cash flow dR_t *plus* the cash flow from financing activities $\dfrac{dI_t}{p} - dD_t$, that is, the cash received from issuing securities minus the cash paid as dividends. Using (4), we rewrite (5) in the form

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 + \frac{I_t}{p} - D_t. \tag{6}$$

Equation (6) is an accounting identity that specifies the relationships between cash reserves, profitability prospects, cumulative issuance and cumulative dividend. Its properties are instrumental for solving explicitly the shareholders' problem. Observe that since ϕ is an increasing function, by holding the cash reserves X_t fixed and the cumulative issuance I_t fixed, the higher the cumulative dividend D_t is, the larger the profitability prospects Y_t. Thus, Equation (6) shows a positive relationship between cumulative dividend and profitability prospects, all else being equal. Similarly, there is a negative relationship between the cumulative issuance and the profitability prospects, all else being equal. Finally, by holding the cumulative dividend and the cumulative issuance fixed, the cumulative cash reserves at t depend on the profitability prospects at t but do not depend on the time elapsed up to t. This property of time-invariance follows from two assumptions. First, because $Y \in \{-\mu, \mu\}$, (4) shows a time-invariant relationship between cash flows and profitability prospects.[11] Second, because the cash inside the firm is not remunerated, there is no additional term in dt in the cumulative cash reserve process (5).[12]

[10] Agency costs of free cash flow can be notably important in the innovation sector due, for instance, to moral hazard between the inventor and financiers. See, e.g., Allen and Michaely (2003), DeAngelo, DeAngelo and Skinner (2008), and Hall and Lerner (2010) for insightful surveys of the literature.

[11] For instance, if $Y \in \{\underline{\mu}, \bar{\mu}\}$ with $\underline{\mu} < 0 < \bar{\mu}$, then (4) writes in the form $dR_t = d\phi(Y_t) + \frac{\underline{\mu}+\bar{\mu}}{2}\, dt$, yielding a time-dependant relationship between cash flows and profitability prospects. Within this setting, see Gryglewicz (2011) for a learning model with no costs of holding cash, and De Angelis (2020) for a learning model with infinite issuance costs.

[12] In particular, if cash reserves earn a rate of interest $r - \lambda$ where $\lambda \in (0, r]$ represents a carry cost of liquidity, then the additional term $(r - \lambda)\, dt$ appears in (5). Thus, we consider in our study the case $\lambda = r$.

The firm ceases its activity for two reasons: (i) the firm cannot meet its short-term operating costs by issuing new shares or by drawing cash from its reserves, and/or (ii) shareholders strategically decide to liquidate because the profitability prospects are not high enough. Thus, given a policy (I, D), equation (5) represents the dynamics of the cash reserves up to the liquidation time τ_0 defined as

$$\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}.$$

For current cash reserves $x \in [0, \infty)$ and current profitability prospects $y \in (-\mu, \mu)$, the value of the firm corresponds to the expected present value of all future dividends minus the expected present value of all future gross issuance proceeds

$$\mathcal{V}(x, y; I, D) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t - dI_t)\right], \tag{7}$$

with $D_{\tau_0} - D_{\tau_0^-} = \max(X_{\tau_0^-}, 0)$. The case $D_{\tau_0} - D_{\tau_0^-} = X_{\tau_0^-} > 0$ corresponds to a strategic liquidation.[13] The shareholders' problem is to find the optimal value function defined as the supremum of (7) over all admissible issuance and dividend policies

$$V^*(x, y) = \sup_{(I,D)\in\mathcal{A}} \mathcal{V}(x, y; I, D), \tag{8}$$

where $\mathcal{A}$ is the set of policies (I, D) such that the associated cash reserves $\{X_t, \mathcal{F}_t^R; t \geq 0\}$ satisfy $X_t \geq 0$, $e^{-rt}X_t$ integrable and, $\lim_{t\to\infty} \mathbb{E}[e^{-rt}X_t] = 0$.

3 Benchmarks

Our model with learning nests two polar cases which themselves are of interest: (i) the case where shareholders face only a profitability concern and, (ii) the case where the shareholders face only a liquidity concern.

3.1 First-best benchmark

Let us consider that shareholders do not observe the actual profitability but face no frictions, that is $p = 1$ in (5). In this framework, accumulating cash does not bring any benefit. The firm's value $\hat{V}$ is equal to the sum of current cash reserves plus the option value to liquidate the firm[14]:

$$\hat{V}(x, y) \equiv x + \sup_{\tau\in\mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs} Y_s ds\right]. \tag{9}$$

An optimal policy is to distribute all of firm's initial cash reserves x as a special payment at date 0, to hold no cash beyond that time, and to liquidate when the profitability prospects hit the so-called first best liquidation threshold $\hat{y}$, which can be explicitly computed. The shareholders' value function V^* in (8) is bounded above by the first-best benchmark $\hat{V}$. To summarize,

[13] In that case, shareholders take the remaining cash reserves $X_{\tau_0^-} > 0$ as dividends.

[14] In (9), we denote by $\mathcal{T}^R$ the set of $\mathcal{F}^R$-stopping times.

Proposition 3.1 *Suppose that shareholders face only a profitability concern then,*

(i) The value of the firm, $\hat{V}(x,y)$, is an increasing and convex function of the level y of the profitability prospects.

(ii) Distributing all initial cash reserves x at time 0, holding no cash beyond that time, and liquidating at the stopping time $\hat{\tau} = \inf\{t \geq 0 \,:\, Y_t = \hat{y}\}$ with $\hat{y} = \frac{-\mu}{1-2\gamma} < 0$ where γ is the negative root of the equation $x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0$, is an optimal policy.

(iii) $V^*(x,y) \leq \hat{V}(x,y)$ for any $(x,y) \in [0,\infty) \times (-\mu,\mu)$.

It follows from assertion (iii) that, if the profitability prospects are lower than the first-best liquidation threshold $\hat{y}$, then the firm is liquidated regardless of the amount of cash within the firm. Formally, $V^*(x,y) = \hat{V}(x,y) = x$ for all $x \geq 0$ and $y \leq \hat{y}$.

3.2 Complete information benchmark

Another useful benchmark is the complete information setting in which shareholders face only a liquidity concern. This corresponds to the case where $y = -\mu$ or $y = \mu$ with $\mu > 0$ in the main problem (8).[15] We shall denote by $V_{-\mu}(x)$ and $V_\mu(x)$ the associated values of the firm.

When $y = -\mu$, the firm's profitability is negative, and it is optimal for shareholders to take the initial cash reserves and to liquidate the firm at time $t = 0$. We have that $V_{-\mu}(x) = x$, $\forall x \geq 0$. When $y = \mu$, the dynamics of the cash reserve process take the form

$$dX_t \;=\; \mu\, dt + \sigma\, dB_t \,+\, \frac{dI_t}{p} \,-\, dD_t.$$

and we revert to a classic case studied by many authors.[16] The value of the firm, $V_\mu(x)$, is an increasing and concave function of the level x of its cash reserves. The concavity of the value function reflects the fact that the marginal value of cash, $V_\mu'(x)$, is decreasing in the level of cash within the firm. It is strictly greater than one up to $x_\mu = \inf\{x > 0 \,|\, V_\mu'(x) = 1\}$ which corresponds to the firm's cash target level at which dividends are paid. If cash holdings x exceed x_μ, the firm places no premium on internal funds, and it is optimal to make a lump sum payment $x - x_\mu$ to shareholders. Accordingly, $V_\mu(x) = x - x_\mu + V_\mu(x_\mu)$ for any $x \geq x_\mu$.

Because external financing is costly, it is optimal to postpone the issuance of new shares for as long as possible: equity issuance only takes place whenever cash reserves are depleted and occurs if and only if the cost of issuance is not too high. Specifically, there exists a threshold $\overline{p}$ such that there is equity issuance every time $X_t = 0$ if and only if $p < \overline{p}$.[17] In that case, the marginal benefit, $V_\mu'(0)$, is equal to the proportional issuance cost, p. Given

[15] Precisely, if $Y_0 = \mu$ (resp. $Y_0 = -\mu$), then $Y_t = \mu$ a.s (resp. $Y_t = -\mu$ a.s) and, for any $Y_0 \in (-\mu,\mu)$, $\inf\{t \geq 0 \,|\, Y_t \notin (-\mu,\mu)\} = \infty$, $\mathbb{P}$-almost surely (it is not possible to learn the true value of profitability Y in finite time).

[16] See, for instance, the textbook by Moreno-Bromberg and Rochet (2018).

[17] The analysis yields that $\overline{p} = \overline{V}'_\mu(0)$, where $\overline{V}_\mu$ corresponds to the value of the firm if issuance of new shares is not allowed (see the Appendix).

that the value of the firm is concave in x, one obtains $V_\mu'(x) < p$ for $x > 0$. This means that it is indeed never optimal to issue new shares before cash reserves are depleted. Finally, the optimal issuance strategy induces a reflection at level zero of the cash reserve process so that infinitesimal amounts of new equity are issued every time $X_t = 0$.

The following proposition summarizes these standard results, first established in Lokka and Zervos (2008) and then used and generalized in several studies, especially in DMRV (2011) and Bolton, Chen and Wang (2011). We provide a rigorous statement in the Appendix, useful for the analysis of our model.

Proposition 3.2 *Suppose that shareholders face only a liquidity concern, so $y = \mu$. Then, the value of the firm, $V_\mu(x)$, is an increasing and concave function of the level x of its cash reserves. Any excess of cash over the dividend boundary x_μ is paid out to shareholders, so the firm's payout ratio is* 100%. *Furthermore,*

(i) If issuance costs are high such that $p \geq \overline{p}$, then it is never optimal to issue new equities, and the firm is liquidated as soon as it runs out of cash.

(ii) If issuance costs are low such that $p < \overline{p}$, then equity issuance takes place whenever the firm runs out of cash, so that the cash reserve process is reflected back whenever it hits 0*, and the firm is never liquidated.*

In the complete information benchmark, when $p < \overline{p}$, firms are never liquidated and the use of cash is simply to delay costly issuances. Things are more intricate in our model because shareholders face also the risk of running a project that is not profitable. A decrease of cash also decreases the belief about the actual profitability of the firm's project and liquidation can be optimal. In contrast to the complete information benchmark, when $p < \overline{p}$, we will see that firms either optimally issue new equity or are liquidated. We will show that learning about profitability dramatically impacts Proposition 3.2: the cash target level, the decision to issue new shares, the payout ratio, the volatility of the firm will be functions of the current profitability prospects that we are able to fully characterize.

4 Model solution

The next section is heuristic and leads to a variational system that should satisfy the value of the firm V^* given by (8).

4.1 Heuristic discussion

Taking the cash reserve and liquidating is an admissible policy, thus the value function V^* satisfies the inequality $V^*(x, y) \geq x$ for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$. From Proposition 3.1, $V^*(x, y) = x$ for all $y \leq \hat{y}$, where $\hat{y}$ is the first-best liquidation threshold. To proceed further, we assume in this section that V^* is as smooth as necessary, and we derive some properties that V^* should satisfy.

Dynamic programming. Let us fix some pair $(x, y) \in (0, \infty) \times (-\mu, \mu)$. Let us consider the policy that consists of abstaining from issuing new shares and paying dividends for $t \wedge \tau_0$

units of time and, then, in applying the optimal policy associated with the resulting couple $(x + \int_0^{t\wedge\tau_0} Y_s\,ds + \sigma dB_s,\ y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2)dB_s)$, implied by dynamics (2) and (3). This policy must yield no more than the optimal policy:

$$\begin{aligned} 0 &\geq \mathbb{E}\left[e^{-r(t\wedge\tau_0)}V^*\left(x + \int_0^{t\wedge\tau_0} Y_s\,ds + \sigma dB_s, y + \int_0^{t\wedge\tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2)dB_s\right)\right] - V^*(x,y) \\ &= \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\left(\mathcal{L}V^*(X_s,Y_s) - rV^*(X_s,Y_s)\right)ds\right]. \end{aligned} \tag{10}$$

The last equality follows from Itô's formula, where $\mathcal{L}$ denotes the partial differential operator defined by

$$\mathcal{L}V(x,y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (\mu^2 - y^2)V_{xy} + yV_x.$$

Letting t go to zero in (10) yields

$$\mathcal{L}V^*(x,y) - rV^*(x,y) \leq 0$$

for all $(x,y) \in (0,\infty) \times (-\mu,\mu)$.

Dividend boundary. The intuition that underlies the complete information benchmark applies: fix some $(x,y) \in (0,\infty) \times (-\mu,\mu)$; the policy that consists of making a payment $\varepsilon \in (0,x)$, and then immediately executing the optimal policy associated with cash reserves $x - \varepsilon$ must yield no more than the optimal policy. That is, $V^*(x,y) \geq V^*(x-\varepsilon,y) + \varepsilon$. Subtracting $V^*(x-\varepsilon,y)$ from both sides of this inequality, dividing through by ε and letting ε approach 0 yield

$$V_x^*(x,y) \geq 1$$

for all $(x,y) \in (0,\infty) \times (-\mu,\mu)$. It is expected that the inequality $V_x^*(x,y) > 1$ holds for any $x \in (0, b^*(y))$, where $b^*(y) = \inf\{x,\ V_x^*(x,y) = 1\} > 0$. Intuitively, for any fixed profitability prospects y, any excess of cash above $b^*(y)$ should be paid out. Therefore, the optimal cash policy should not be characterized by a constant threshold, as in the complete information benchmark, but rather by a dividend boundary $y \longrightarrow b^*(y)$.

Issuance policy. If it is never optimal to issue new shares when the firm's profitability is known and equal to μ, then it should also never be optimal to issue new shares in the incomplete information setting. Thus, if $p \geq \overline{p}$, we expect that the firm is liquidated when it runs out of cash. If $1 < p < \overline{p}$, the logic of the complete information benchmark applies again: if there is any issuance activity, this must be when cash reserves drop down to zero to avoid liquidation. In such a situation, the marginal value of cash should be equal to the proportional issuance cost p, formally, $V_x^*(0,y) = p$. Intuitively, this latter equality should require that the profitability prospects when the cash reserves are depleted are sufficiently high. Accordingly, we conjecture the existence of an (endogenous) threshold y_i^* such that $V_x^*(0,y) = p$ for any $y \geq y_i^*$, whereas $V^*(0,y) = 0$ for any $y \leq y_i^*$. In this latter case, the profitability prospects are too low with regard to the cost of external financing, and the firm defaults when the cash reserves are depleted.[18]

[18] The subscript "i" in y_i^* and throughout the paper stands for *issuance*.

Convergence toward the complete information benchmark. Finally, we expect that when shareholders are increasingly confident that the profitability of the firm is μ, the value of the firm tends to the one derived in the complete information benchmark. We should have for all $x \geq 0$

$$\lim_{y \longrightarrow \mu} V^*(x, y) = V_\mu(x). \tag{11}$$

One thus expects that the value function of the shareholders' problem (8) satisfies on $[0, \infty) \times (-\mu, \mu)$ the HJB Equation

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0,$$

the condition $\max(-V(0, y), V_x(0, y) - p) = 0$ on $y \in (-\mu, \mu)$, and the limit condition (11). We rephrase this guess in terms of the following variational system: find a smooth function V, a constant $y_i \in (-\mu, \mu)$ and a positive function b continuously differentiable almost everywhere over $(-\mu, \mu)$ that solve

$$\begin{aligned}
\mathcal{L}V(x, y) - rV(x, y) &= 0 \quad \text{on the domain } \{(x, y),\ 0 < x < b(y),\ -\mu < y < \mu\}, && (12)\\
V(0, y) &= 0 \quad \forall y \in (-\mu, y_i], && (13)\\
V_x(0, y) &= p \quad \forall y \in [y_i, \mu), && (14)\\
V_x(x, y) &= 1, \text{ for } x \geq b(y), && (15)\\
V_{xy}(b(y), y) &= 0, && (16)\\
\lim_{y \longrightarrow \mu} V(x, y) &= V_\mu(x) \ \ \forall x \geq 0. && (17)
\end{aligned}$$

Condition (15) and (16) follow from the heuristics on the dividend boundary b^* together with the "principle of smooth-fit", which postulates sufficient smoothness of the value function of stochastic control problem. We will see that the system (12), (13), (14), (15), (16) has an uncountable set of solutions that are twice continuous differentiable on any open set in $(0, \infty) \times (-\mu, \mu)$. It will turn out that, the limit condition (17) pins down the only solution of system (12)-(16) that is twice continuous differentiable on any open set in $(0, \infty) \times (-\mu, \mu)$ *and* has bounded first derivatives. This result allows us to develop a verification procedure and to show that the unique solution (V, y_i^*, b^*) to the system (12)-(17) coincides with the value function V^* of the shareholders' problem, and that y_i^* and b^* characterize the optimal issuance and dividend policies. We obtain a closed form expression for V given y_i^* and b^*, and show that three highly non linear equations characterize the threshold y_i^* and the function b^*. A rich set of implications follows.

4.2 Solution to the shareholders' problem

In this section, we focus on the case $1 < p < \overline{p}$ and explain how we solve the system (12)-(17).[19] Then, we state our main result. Our analysis relies on a simple change of variable, which will be proved very useful for both the mathematical treatment and the economic analysis of the model.

As we explained, it follows from (6) that, holding the cumulative dividend and the cumulative issuance fixed, there is a time-invariant relationship between the cash reserves and

[19] We refer the reader to the Appendix for the complete analysis.

the profitability prospects. That is, as long as the controls I and D are not activated, the process $Z = \{Z_t; t \geq 0\}$ with

$$Z_t \equiv \phi(Y_t) - X_t \tag{18}$$

remains constant. The change of variable (18) allows us to restate problem (12)-(17) in the (z, y)-space and to solve it analytically.[20] The change of variable (18) also provides new economic insights that we comment on below.

To develop the intuition, let us consider some admissible issuance and dividend policies I and D leading to cash reserve process

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 - D_t + \frac{I_t}{p}.$$

It follows that

$$Z_t = \phi(Y_t) - X_t = D_t - \frac{I_t}{p} + (\phi(Y_0) - X_0).$$

The process Z_t corresponds to the cash outflows from financing activities, $D_t - \frac{I_t}{p}$, corrected for the initial amount $\phi(Y_0) - X_0$. The process Z_t increases whenever the firm reaches a cash target level and decreases whenever the firm issues new shares. It measures the performance record of the firm at time t and defines a one-to-one mapping between profitability prospects and cash reserves that holds true as long as the firm neither pays dividends nor issues new securities.

It is worth noting that the current cash outflow from financing activities, $D_t - I_t/p$, is not a sufficient statistic to define the performance of the firm at date t. The firm's performance at date t also depends on the initial profitability prospects Y_0 through the relation $\phi(Y_0) - X_0$. The initial profitability prospects Y_0 are not directly observable and follow, for instance, from a specific analysis by financial analysts of the relevance of the firm's project at the early stage of the firm's life. Thus, the performance of the firm is defined in light of the initial assessment of the profitability prospects. We will see that the firm's performance process Z indicates whether the firm can optimally issue new shares if needed.

Using the change of variable (18), we define

$$U(z, y) \equiv V(\phi(y) - z, y),$$

and we restate problem (12)-(17) in the (z, y)-space. In the (z, y)-space, the partial differential equation

$$\mathcal{L}V(x, y) - rV(x, y) = 0$$

becomes

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0. \tag{19}$$

The solution to (19) writes in the form

$$U(z, y) = A(z)h_1(y) + B(z)h_2(y), \tag{20}$$

[20]The property that one of the two state variables remains constant over the inaction region arises naturally in stopping problems involving the running maximum of a diffusion (e.g Peskir and Shiryaev (2006)) and in the so called finite fuel control problems (e.g Karatzas, Ocone, Wang, Zervos (2000)). Our study suggests that some Bayesian adaptive singular control problems can be framed to satisfy this property as well, leading to full characterization of optimal controls.

where $h_1(y) = (\mu + y)^\gamma (\mu - y)^{1-\gamma}$ and $h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$, with γ being the negative root of the equation $x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0$, are two fundamental solutions to the equation

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 u''(y) - ru(y) = 0, \tag{21}$$

and where A and B are two functions that boundary conditions will specify. Thus, because the process Z remains constant as long as controls are not activated, studying the shareholder's value function in the inaction region leads, in the (z, y)-space, to solve the ordinary differential equation (21) and not a partial differential equation as it is the case in the (x, y)-space.

The condition $V(0, y) = 0$ for all $y \leq y_i$ becomes

$$U(\phi(y), y) = U(z, \psi(z)) = 0, \text{ for all } z \leq z_i,$$

where $z_i \equiv \phi(y_i)$ and where $\psi(z) \equiv \phi^{-1}(z) = \mu \frac{e^{\frac{2\mu}{\sigma^2}z} - 1}{e^{\frac{2\mu}{\sigma^2}z} + 1}$. Thus, for a given performance z, the real number $\psi(z) \in (-\mu, +\mu)$ corresponds to the profitability prospects when cash reserves are depleted, that is, when $x = 0$.

The condition $V_x(0, y) = p$ becomes

$$U_z(z, \psi(z)) = -p,$$

which holds for any $z \geq z_i$. In other words, the firm issues new shares when cash reserves are depleted only if its performance is higher than z_i (equivalently, if the profitability prospects when the cash reserves are depleted are higher than y_i). We call z_i, the market threshold.

The condition $V_x(b(y), y) = 1$ becomes

$$U_z((\phi - b)(y), y) = U_z(z, k(z)) = -1,$$

where $k(z) \equiv \inf\{y \,|\, (\phi - b)(y) = z\}$. Thus, according to the change of variable (18), $k(z)$ corresponds to the profitability prospects at the cash target level $b(k(z)) = \phi(k(z)) - z$. Therefore, in our two-dimensional setting, each performance z defines a cash target level $b(k(z))$, the set of which forms the dividend boundary. We will see that the function $\phi - b$ is invertible, so $k(z) = (\phi - b)^{-1}(z)$. We will prove that k is an increasing function. Intuitively, the higher the firm's performance is, the higher the profitability prospects at the cash target level.

The contact condition $V_{xy}(b(y), y) = 0$ becomes

$$U_{zy}((\phi - b)(y), y) = U_{zy}(z, k(z)) = 0.$$

The convergence condition to the complete information benchmark, $\lim_{y \longrightarrow \mu} V(x, y) = V_\mu(x)$, becomes

$$\lim_{z \longrightarrow \infty} U(z, \psi(x + z)) = V_\mu(x).$$

Indeed, the change of variable (18) leads to $y = \psi(x+z)$ and, in turn, $V(x, \psi(x+z)) = U(z, \psi(x+z))$.

Overall, the free boundary problem (12)-(17) writes in the (z,y)-space in the following form: find a function U, a constant z_i, and a function k that solve the variational system

$$\begin{aligned}
\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z,y) - rU(z,y) &= 0 \quad \text{on the domain} \\
&\qquad \{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}, && (22)\\
U(z,\psi(z)) &= 0 \quad \forall z \leq z_i, && (23)\\
U_z(z,\psi(z)) &= -p \quad \forall z \geq z_i, && (24)\\
U_z(z,y) &= -1, \text{ for } k(z) \leq y, && (25)\\
U_{zy}(z,k(z)) &= 0. && (26)\\
\lim_{z \longrightarrow \infty} U(z, \psi(x+z)) &= V_\mu(x). && (27)
\end{aligned}$$

We find the unique solution (U, z_i^*, k^*) to the system (22)-(27). We obtain the solution (V, y_i^*, b^*) to the system (12)-(17) through the relations $U(z,y) = V(\phi(y) - z, y)$, $k^*(z) = (\phi - b^*)^{-1}(z)$ and, $z_i^* = \phi(y_i^*)$. We show that this solution coincides with the value function of the shareholders' problem and we characterize the optimal strategies. We establish these results in a series of Propositions in the Appendix. To ease its reading, we present informally below the main ideas that underly the proofs.

Let us consider a solution (U, z_i, k), if any, to the system (22)-(26). Note that we do not take into account for the moment the condition (27). We obtain from (20), (24), (25), (26) that the function k satisfies for $z \geq z_i$

$$-h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p = 0. \tag{28}$$

Proposition 8.11, assertion (i) establishes that (28) defines over $[z_i, +\infty)$ a unique continuously differentiable increasing function k. Proposition 8.11, assertion (ii) characterizes the unique z_i^* that satisfies (27). We obtain from (20), (23), (25), (26) that, for $z \leq z_i^*$, the function k satisfies an ordinary differential equation, which terminal condition ensures the continuity of k at z_i^*. This uniquely defines a function k^* associated to the system (22)-(27). Getting this ordinary differential equation is quite involved and requires the analysis of the case where shareholders are not allowed to issue new shares, which we develop as a preliminary step in section 8.3.1. Equation (95) in Proposition 8.11, provides the full description of the function k^* which therefore involves three highly non linear equations.[21] Proposition 8.12 establishes that the above procedure determines a unique solution (V, y_i^*, b^*) to the system (12)-(17). The function V has a closed form expression given $z_i^* = \phi(y_i^*)$ and $k^* = (\phi - b^*)^{-1}$. The function b^* is not differentiable at $k^*(\phi(y_i^*))$. We check that the value function V solution to the system (12)-(17) coincides with the value function V^* solution to the shareholder's problem. To this end, Proposition 8.13 shows that the solution to (12)-(17)

[21]Precisely, Equation (28), Equation (94) in Proposition 8.11 that characterizes z_i^* and, the ordinary differential equation studied in Lemma 8.5 with the terminal terminal condition that ensures the continuity of k^* at z_i^* as stated in assertion (iii) of Proposition 8.11.

satisfies the verification Lemma 8.9. Notably, Proposition 8.13 shows that a solution V to the system (12)-(16) has bounded first derivatives if and only if V converges to the value function of the complete information benchmark, that is, if and only if condition (17) holds. Thus, a remarkable feature of our two-dimensional control problem is that the condition (17) pins down the unique solution to the system (12)-(16) which coincides with the value function of the shareholder's problem. Proposition 8.14 describes the optimal issuance and dividend policies. Finally, Proposition 8.15 establishes that the dividend boundary function b^* is increasing in the profitability prospects when issuance costs are large (in which case $y_i^* = \mu$), and nonmonotonic in the profitability prospects when issuance costs are low ($p < \overline{p}$). It reaches its maximum at $k^*(z_i^*) = k^*(\phi(y_i^*))$

The next Theorem summarizes our results. Its statement is in mirror of Proposition 3.2.

Theorem 4.1 *The value of the firm V^* given by (8) coincides with the unique solution (V, y_i^*, b^*) to the system (12)-(17). The threshold y_i^* corresponds to the required profitability prospects above which the firm optimally issues new shares when the cash reserves are depleted and below which the firm is liquidated when it runs out of cash. Payments are made whenever cash reserves hit the dividend boundary function b^*. The function b^* is continuous over $[-\mu, \mu]$ and satisfies $b^*(y) = 0$ for $y \leq y^*$ and $b^*(\mu) = x_\mu$. The threshold x_μ is the constant dividend boundary of the complete information benchmark. The threshold $y^* = max\{y \in (-\mu, \mu) \,|\, y = b^{*-1}(0)\} > \hat{y}$ corresponds to the minimum profitability prospects required by shareholders to run the project. The optimal cash reserve process is reflected along the function b^* in a horizontal direction on the (x, y)-plane. Furthermore:*

(i) If the proportional issuance cost p satisfies $p \geq \overline{p}$,

- *$y_i^* = \mu$, so it is never optimal to recapitalize the firm. The firm is liquidated when it runs out of cash, $V^*(0, y) = 0$ for all $y \in (-\mu, \mu)$.*

- *The dividend b^* is continuously increasing and differentiable.*

(ii) If the proportional issuance cost p satisfies $1 < p < \overline{p}$,

- *$y_i^* \in (y^*, \mu)$, so that equity issuance takes place whenever the firm runs out of cash if and only if the profitability prospects are greater than the threshold y_i^*. At issuance dates, the optimal cash reserve process is reflected in a horizontal direction on the (x, y)-plane, the marginal value of cash $V_x^*(0, y)$ is equal to the issuance cost p, and $V^*(0, y) > 0$ for all $y > y_i^*$.*

- *The dividend boundary b^* is increasing for $y \leq k^*(z_i^*)$ and decreasing for $y \geq k^*(z_i^*)$, where $z_i^* = \phi(y_i^*)$ and $k^* = (\phi - b^*)^{-1}$. The maximum cash target level, $b^*(k^*(z_i^*))$, satisfies $b^*(k^*(z_i^*)) > x_\mu$. The dividend boundary is not differentiable at $k^*(z_i^*)$.*

Theorem 4.1 delivers several results. In our model, uncertainty about the firm's profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its profitability. Two opposite effects are at work. First, a positive shock to earnings increases profitability prospects and may induce the firm's management to lower cash target levels because of the cost of accumulating cash. Second, a firm has more to lose from liquidity constraints when profitability prospects are high than when they are low. This may induce the firm's management to accumulate more cash when profitability prospects increase. Theorem 4.1 shows that the second effect dominates when the firm cannot tap the capital markets, while the first effect dominates when the firm can optimally resort to the capital markets. As a consequence, the dividend boundary b^* is increasing in the profitability prospects for high issuance costs (when $p \geq \overline{p}$) and is non-monotonic in the profitability prospects for low issuance costs (when $p < \overline{p}$). In the latter case, the corporate cash target level is at its highest value at the market threshold z_i^*. At that moment, the firm has a higher cash target level than what would have been optimal in a complete information setting with profitability μ that is, $b^*(k^*(z_i^*)) > x_\mu$. Another salient feature that we comment on in the next section is the non-differentiability of b^* at $k^*(z_i^*)$.

When $p < \overline{p}$, if the profitability prospects when cash reserves are depleted are above y_i^*, then the firm issues new shares, whereas the firm is liquidated if profitability prospects are lower than y_i^*. Thus, in our learning model, profitability issues create liquidity issues, and ultimately, the firm is liquidated for liquidity reasons.

Finally, let us note that the profitability prospects y^* required to run the project are strictly larger than the first-best liquidation threshold $\hat{y}$. A negative exogenous shock that leads to profitability prospects below y^* triggers liquidation even if cash reserves are abundant.

In the next section, we alleviate the notations and we write V, y_i, z_i, k and b instead of V^*, y_i^*, z_i^*, k^* and b^*. Theorem 4.1 allows this shortcut since it establishes that the solution to the system (12)-(17) coincides with the value function of the shareholders problem and characterizes optimal policies.

5 Model Analysis

Our analytical formulae allow simple numerical illustrations. Figure 1 plots on the (x, y)-plane the dividend boundary b and the curves $z = \phi(y) - x$ that link cash reserves and profitability prospects for a firm's different performance levels, z. It illustrates the joint dynamics of cash and profitability prospects. The parameters r, μ and σ are annualized; σ and μ are expressed in millions of dollars.

The joint dynamics of cash and profitability prospects: an illustration based on Figure 1. Assume that, at date $t = 0$, the cash reserves X_0 and the profitability prospects Y_0 satisfy the equation $Z_0 = \phi(Y_0) - X_0$ with $Z_0 = -0.40$ such that the pair (X_0, Y_0) is on the long dashed curve in Figure 1. The amount $Z_0 = -0.40$ corresponds to the initial value of the performance process $Z_t = \phi(Y_t) - X_t = Z_0 + D_t - I_t/p$. As long as there are neither payments nor issuances, $D_t = I_t = 0$, the performance process Z remains constant. Therefore, the two-dimensional process $(X_t,\ Y_t)$ satisfies $\phi(Y_t) - X_t = Z_0$ and thus evolves on the long dashed curve. If the cash reserves increase to the point of exceeding the dividend boundary b (solid curve), cash is paid out, the cash reserve process is reflected back in the horizontal direction on the (x, y)-plane, and the performance process increases. If performance records accumulate, the process Z_t will eventually increase to the value $z = 0$, so that the process (X_t, Y_t) will satisfy $\phi(Y_t) - X_t = 0$ and thus will evolve on the short dashed curve.

Consider now that the cash reserves decrease after a series of negative shocks on cash

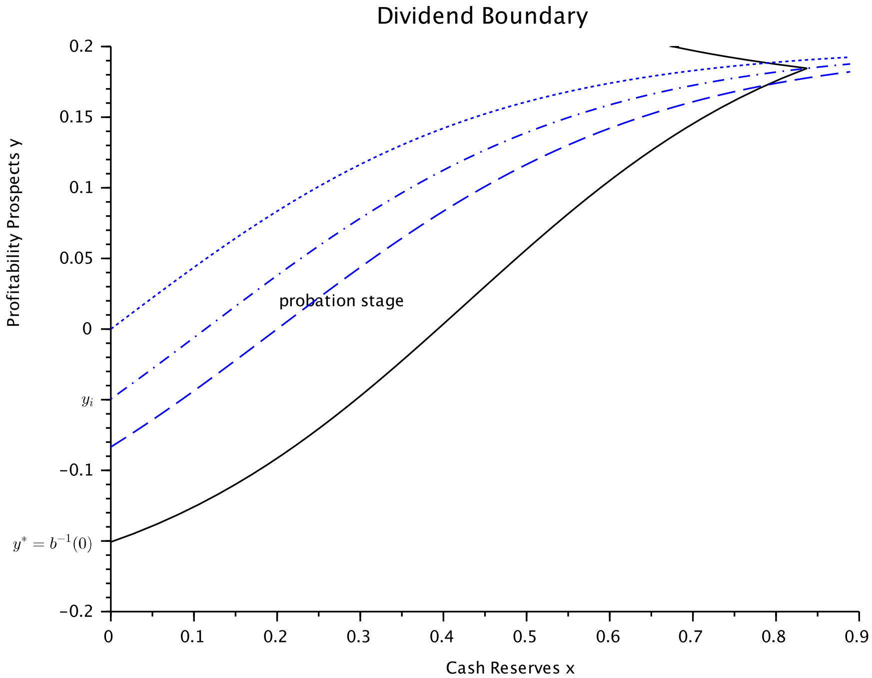

Figure 1: The dividend boundary b (solid curve) and three performance level curves $z = \phi(y) - x$ with $z = -0.40$ (long dashed curve), $z = z_i$ (dash-dotted curve), $z = 0$ (short dashed curve). The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 3$. For those parameters, $y_i = -0.049$, $z_i = -0.1135$, $k(z_i) = k(\phi(y_i)) = 0.1845$, $b(k(z_i)) = 0.8374$ and $x_\mu = 0.6762$. Accordingly, $z_i = \phi(y_i)$ corresponds to the minimum level of performance that allows the firm to issue new shares when cash reserves are depleted, and $k(z_i) = (\phi - b)^{-1}(z_i)$ corresponds to the profitability prospects at the cash target level when the performance is z_i.

flows to hit zero. Shareholders then decide whether to issue new shares. They do so whenever the profitability prospects are larger than y_i or, equivalently, whenever the performance of the firm is above the market threshold z_i. When the firm issues new shares, the cash reserve process is reflected back in the horizontal direction on the (x, y)-plane, and the performance process decreases accordingly. If, as time passes, cumulative issuances become too large, the process (X_t, Y_t) will eventually evolve on a constant-performance curve below the dash-dotted curve $z_i = \phi(y) - x$. The firm then runs the risk of being liquidated because, for such a level of performance, the profitability prospects are too low compared to the financing cost p to allow shareholders to issue new shares when cash reserves are depleted.

Thus, our model results in a two-stage life cycle for the firm: a “probation” stage (below the dash-dotted curve) in which the firm cannot resort to the market to meet short-term obligations and a “mature” stage (above the dash-dotted curve) in which the firm can raise new funds from the market if needed. In the probation stage, cash target levels are increasing in the profitability prospects. In the mature stage, the precautionary motive for holding cash weakens, and cash target levels decline when the profitability prospects increase. Therefore, the dividend boundary b attains its maximum when the firm is at the market threshold z_i. Figure 1 also illustrates that the function b attains its maximum at a kink, meaning that it is non-differentiable at its maximum as stated in Theorem 4.1. We comment further these results in the next paragraph.

Finally, let us observe that, two identical firms with the same cash outflow from financing activities, $D_t - I_t/p > 0$, but with different initial profitability prospects and thus different values for Z_0, for example, $Z_0^1 < z_i < Z_0^2$, can be in drastically different situations when cash reserves are depleted: it can happen that firm 1 is liquidated because $Z_t^1 = Z_0^1 + D_t - I_t/p < z_i$, whereas firm 2 issues new shares because $Z_t^2 = Z_0^2 + D_t - I_t/p \geq z_i$. Thus, the initial profitability prospects, which may result from financial analysts, have lasting effects on the corporate cash policy.

Payout ratios. What does a firm do with a marginal \$1 when it is at its target level of cash? In the complete information benchmark, the firm pays out \$1 as a dividend. This generates the prediction that the payout ratio of a firm is constant and equal to 100%. In our learning model, one expects that the payout ratio is a function of the profitability prospects. Because we provide an analytical characterization of the dividend boundary, we are able to study the payout ratio that our two-dimensional model induces. This is a unique feature of our paper. Specifically, suppose that the firm is at its cash target level $x = b(y)$ and consider what happens after a positive shock to the cash flow. To account for the sign of the change in cash flow over a small period of time h, we consider a Euler approximation of the model,

$$X_h = x + \sigma\sqrt{h}B_1, \quad \text{and} \quad Y_h = y + \frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1,$$

where B_1 is a standard Gaussian variable. Therefore, $X_h - x$, the amount of cash available for distribution at time h, and $X_h - b(Y_h)$, the amount paid out to shareholders at time h, satisfy

$$\begin{aligned} X_h - b(Y_h) = X_h - (b(y) + b'(y)(Y_h - y)) &= X_h - x - b'(y)\frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1 \\ &= P(y)(X_h - x), \end{aligned}$$

with

$$P(y) = 1 - b'(y)\frac{\mu^2 - y^2}{\sigma^2}. \tag{29}$$

The function $P(y)$ indicates, for profitability prospects y, the percentage of each dollar earned above the dividend boundary that is distributed to shareholders in the form of cash. It corresponds to the firm's payout ratio. Therefore, in our two-dimensional model, a dividend decision is made when cash reserves reach the cash target level $b(y)$. Shareholders then receive $P(y)$ percent of the cash above $b(y)$ and reinvest the complement into the firm.

Formula (29) relates the payout ratio to the derivative of the dividend boundary b and yields the following result. In the probation stage, after a new performance record the firm increases its cash target level, (the derivative of the dividend boundary is positive). It follows from (29) that the payout ratio is lower than 100%. In the mature stage, the firm can raise funds from the market. It decreases its cash target level after a performance record (the derivative of the dividend boundary is negative). The firm has a payout ratio larger than 100% meaning that the firm dis-saves and uses its reserves to pay more dividends than its last profit. The decision to decrease cash target levels yields a discontinuity in the payout ratio at $k(z_i)$ that originates from the non-differentiability of b at $k(z_i)$. This break reflects the change in the cash management policy when changing between corporate life-cycle stages. When its performance level allows the firm to enter the mature stage, the cost of holding cash becomes prominent, leading to a change in the payout policy. It is worth noting that the jump in the payout ratio is not related to a jump in firm value, which continuously evolves as a function of cash reserves and profitability prospects. Let us also observe that because the dividend boundary is non-monotone, the cash target level is not a sufficient statistic to infer the payout ratio. We must know what stage the firm is in to deduce the payout ratio from cash target levels. Finally, the payout ratio tends toward 100% when y tends toward μ, reflecting the fact that the model converges to the complete information benchmark.

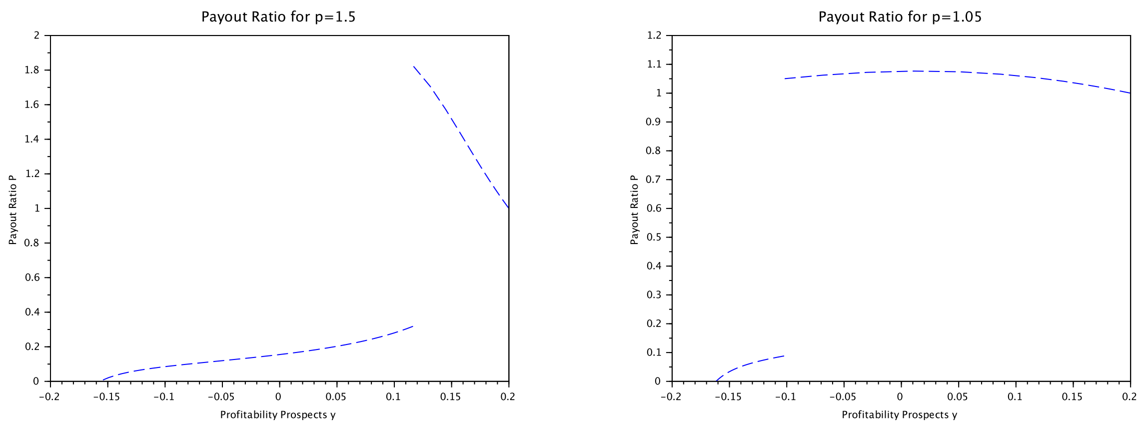

Figure 2: The payout ratio as a function of profitability prospects. (Left) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.05$

Numerical simulations provide additional insights. Figure 2 is representative of our numerical simulations and plots the payout ratio for proportional issuance costs $p = 1.5$ and $p = 1.05$ (with other baseline parameters remaining the same). Our model suggests that a cash-constrained firm which profitability is difficult to ascertain pays very little in dividends as long as it cannot tap the market and initiates dividend when it can resort to the market. This effect is even more true when the cost of external financing is high. In particular a firm in a costly issuance environment has a larger payout ratio when it enters into the mature stage than a firm in a cheap issuance environment. The reason is that both the firm's optimal level of cash and required profitability prospects to reach the mature stage increase with the cost of external financing. It follows that a high-cost firm has more slack with which to respond to positive shocks to earnings and initiates high payments (equivalently dis-saves aggressively) in the mature stage to counteract the cost of holding cash.

Relationsip to the empirical literature. Our results are consistent with several empirical findings on which our learning model sheds new light. The study of DeAngelo, DeAngelo and Stulz (2006) suggests that, if well-established firms had not paid dividends as observed, their cash balances would be enormous, thus granting extreme discretion to managers of these firms. They find also that the relation between cash holdings and propensity to pay dividends is ambiguous. Their proposed explanation for this is that cash holdings "could indicate a build-up of excess funds (best suited for distribution) or of resources to fund an abundance of new investments (best suited for retention)". In our model the learning about the actual long term profitability of the project drives the build-up and the decrease of cash holdings. It results that the propensity to pay dividends can be different for the same level of cash holdings. Bulan, Subramanian and Tanlu (2007) document that firms that initiate dividends have increased their profitability and have greater cash reserves. In our model the firm initiates dividends to fit cash holdings with beliefs about the actual profitability. DeAngelo, DeAngelo, and Stulz (2010) document that a SEO reflects the corporate life cycle and that, without the offer proceeds, most firms would run out of cash the year after the SEO. Our model provides this prediction where the optimality of SEO follows from Bayesian learning about the firm's profitability. Dickinson (2011) documents that cash flow patterns are related to the firm life cycle. Faff, Kwok, Podolski and Wong (2016) point out the non-monotonicity of the firm's cash holdings across the corporate life-cycles and question the relevance of cash holdings as a good proxy of the stages of the life-cycle. Drobetz, Halling and Schroder (2015) find evidence of increases in cash holdings in stages of the life cycle where external financing is more difficult and of decreases in cash holdings when the firms move toward maturity. Our model suggests a possible explanation to the above empirical findings on which theoretical models are scarce. Notably, the non-monotonicity of cash target levels that we derive explains that the cash target level per se do not allow indeed to infer the firm's life cycle. More generally, our study suggests that learning models could potentially explain documented facts on the relationship between the corporate life cycle and the dynamics and valuation of cash holdings. Additionally, our model suggests that the initial assessment of the firm's profitability has long lasting consequences on its corporate cash management and matters for evaluating its performance at any future date. This feature is consistent with the important role of specialized intermediaries in the financing of innovation as, for instance, pointed in Kerr and Nanda (2015).

The next paragraph explains the relationship between the value of the firm, its volatility and the volatility of the cash flow that results from our learning model. In doing so, we propose new empirical predictions grounded on the Bayesian learning about the firm's profitability.

Firm dynamics. Our model delivers new insights into the dynamics of firm value across life stages. Let us consider the firm value process between two consecutive dates of issuance decision and payment decision. We saw that for a given performance $z \in [\phi(y^*), \infty)$, payment occurs at time $\tau_z \equiv \{t \geq 0 \,|\, X_t = b(k(z))\}$, and issuance (or liquidation) at time $\tau_0 = \{t \geq 0 \,|\, X_{t^-} = 0\}$. Thus, the firm value process can be written on time interval $[0, \tau_0 \wedge \tau_z]$ as a function of the cash reserve process $\{X_t,\, t \geq 0\}$. We use the change of variable (18) to obtain that $V(x, y) = V(x, \psi(x + z))$, and we denote $W(x, z) \equiv V(x, \psi(x + z))$. Then, by applying Itô's formula to the process $\{W(X_t, z),\ t \geq 0\}$, we easily obtain the following proposition.

Proposition 5.1 *For a given performance, $z \geq \phi(y^*)$, the mapping $x \longrightarrow W(x, z)$ is increasing on $[0, b(k(z))]$. The firm value process $\{W(X_t, z),\ t \geq 0\}$ satisfies, for any*

$t \in [0, \tau_0 \wedge \tau_z]$, *the dynamics*

$$dW(X_t, z) = rW(X_t, z)\, dt \ + \ \sigma\, W_x(X_t, z)\, dB_t,$$

with

$$\sigma W_x(x, z) = \sigma V_x(x, \psi(x+z)) + \sigma \psi'(x+z) V_y(x, \psi(x+z)), \tag{30}$$

so that the volatility of the firm at cash target $b(k(z))$ *satisfies*

$$\sigma\, W_x(b(k(z)), z) = \sigma + \frac{\mu^2 - k(z)^2}{\sigma} V_y(b(k(z)), k(z)). \tag{31}$$

Thus, for a fixed level of performance z, the mapping $x \longrightarrow W(x, z)$ represents the value of the firm as a function of the cash reserves between two consecutive dates of issuance decision and payment decision. Between these two dates, an increase in cash also increases profitability prospects so that the relation $x - \phi(y) = z$ holds true.

Figure 3 depicts, for different issuance costs, the volatility of the firm as a function of the cash reserves with different levels of performance z (Equation (30)). For a z sufficiently above z_i, the volatility of the firm is decreasing in the cash reserves. For z sufficiently below z_i, the volatility of the firm is increasing in the cash reserves. Close to the market threshold (when z is around z_i), neither effects fully dominate and the volatility of the firm is increasing and then decreasing in the cash reserves. Then, noting that the mapping

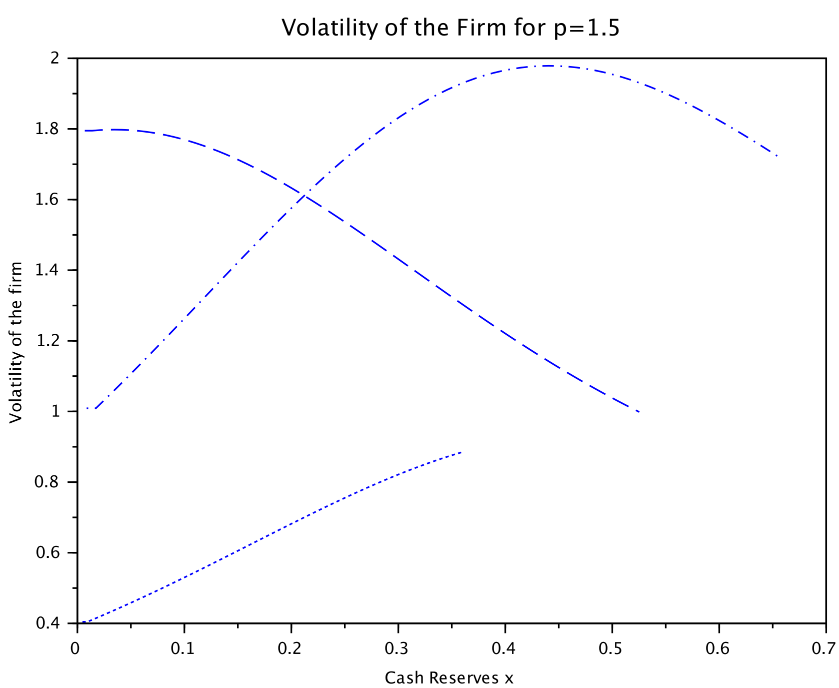

Figure 3: The volatility of the firm as a function of cash reserves. The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$, $z = -0.4$ (dotted curve), $z = z_i$ (dash-dotted curve) and $z = 0$ (dashed curve) and $p = 1.5$.

$x \longrightarrow W(x,z)$ is increasing, we get new testable results on the relationship between the value of the firm and its volatility across firm life-cycle stages. Our model predicts that, for firms in the mature stage, we should observe a negative relationship between firm value and volatility. For firms with performance still far from the market threshold, we should observe a positive relationship between firm value and volatility. The negative relationship between the firm's value and its volatility is a standard feature of corporate cash models with complete information.[22] The positive relationship between the volatility of the firm and its value is a well-known feature of corporate models with deep-pocketed shareholders[23] and holds true in our first-best benchmark. Our study shows that the result still holds for cash-constrained shareholders who learn about profitability and cannot resort to the market to raise new funds. Close to the market threshold z_i, the volatility of the firm is an inverted U-shaped function of the value of the firm, reflecting the two effects. Therefore, our model suggests that the relationships between the value of the firm and its volatility can drastically change in transition phases between life-cycle stages.

In the complete information benchmark, the volatility of the firm, $\sigma V'_\mu(x)$, and the volatility of cash flows, σ, coincide at the cash target level. In our model, because of the second term of the right-hand side of (30) which reflects Bayesian learning, the volatility of the firm is larger than the volatility of cash flows.[24]

Two additional insights on the firm's volatility follow from our model. First, we observe in Figure 3 that the volatility of the firm when cash reserves are depleted is increasing in the performance level z. Thus, the model predicts that the higher the profitability prospects, the higher the volatility of the firm is at issuance dates. Second, at cash target levels, the firm reaches its higher level of volatility when it is at the market threshold z_i. This is consistent with the prediction that the dynamics of cash holdings change drastically when the firm's performance crosses the threshold z_i.

6 Conclusion and Discussion

This paper studies how Bayesian learning about the firm's profitability interacts with the precautionary motive for holding cash. The shareholders' problem takes the form of a two-dimensional control problem that we solve by means of an explicit construction of its value function. This allows a complete and rigorous analysis of the model based on analytical characterizations of optimal issuance and payment policies which is unique to our paper. We show that learning generates a corporate life-cycle with two stages: a "probation stage" where it is never optimal for the firm to issue new shares, and a "mature stage" where the firm resorts to the market whenever needed. The firm's key indicators cash target levels, payout ratios, volatility of the firm feature different properties and relationships across the corporate life cycles.

[22]In the complete information benchmark, the volatility of the firm corresponds to the volatility of the cash flows times the marginal value of cash (that is, $\sigma V'_\mu(x)$ with the notations of Proposition 2).

[23]We refer to the literature initiated by Merton (1973) and Leland (1994).

[24]Technically, at the limit, for $z = \infty$, the model coincides with the complete information benchmark. We show in the Appendix (see Propositions 8.7 and 8.13) that $\lim_{z \longrightarrow \infty} W(x,z) = V_\mu(x)$ and that $\lim_{z \longrightarrow \infty} W_x(x,z) = V'_\mu(x)$. In particular, we have the equality $\sigma \lim_{z \longrightarrow \infty} W_x(b(k(z)),z) = \sigma$.

The dynamics of the cash reserve process (Equation (6)) is instrumental for our analysis. It shows a time-invariant relationship between cash reserves and profitability prospects as long as controls (issuances and payments) are not activated. This property follows from two main assumptions: An arithmetic Brownian motion with unobservable drift $Y \in \{-\mu, \mu\}$ models the cumulative cash flow process and, the cash is not remunerated inside the firm. Introducing cash-carry costs into two-dimensional corporate cash models raises major issues regarding the regularity of the value function, the existence and the characterization of optimal policies. Reppen, Rochet and Soner (2020) are the first to prove the continuity of the value function of the shareholders' problem in a two-dimensional model with complete information and where cash is not remunerated inside the firm. However, similarly to other two-dimensional models, the existence of an optimal payment policy is not proven. We could consider other modeling assumptions on the unobserved profitability provided that the filtering Equation (4) features a time-invariant relationship between cash flows and profitability prospects, in order to obtain an analogous to Equation (6). For instance, we could introduce into the model a geometric Brownian motion with unobserved drift to study permanent shocks to profitability or to cash holdings. Depending on the modeling choices, several corporate models with complete information can serve as benchmarks.[25,26] Exploring this idea could be a topic for future research.

Our assumptions are simple and natural given the objective to study analytically how incomplete information impacts pioneering corporate cash models. We briefly discuss two possible generalizations. We have assumed proportional issuance costs which implies that equity issues occur in infinitisimal amounts. Intuitively, a combination of fixed and proportional issuance costs would imply lumpy equity issuance. Introducing fixed issuance costs in corporate cash model leads to the study of mixed singular/impulse control problems which are difficult even in a complete information setting.[27] In our learning model adding fixed issuance costs does not impact the time-invariant relationship between cash reserves and profitability prospects as long as controls are not activated. The main intuition is unchanged: when profitability prospects are high, the cost of holding cash dominates equity issuance costs while the opposite occurs for low profitability prospects. We therefore expect a corporate life cycle with two stages and a dividend boundary which is non-monotonic in the belief. However, solving explicitly the problem is challenging and worth of interest both from a mathematical and economic point of views: considering fixed issuance costs should lead to an additional free boundary characterizing the gross financing raised by the investor when cash is depleted and should provide more insights on the impact of learning on the amount of cash raised by investors, a question not yet addressed in the literature.

Another natural question is the ability of our model to incorporate investment into the analysis. Models with complete information suggest different ways to consider investment in corporate cash models.[28] In the spirit of Babenko and Tserlukevich (2021), let us consider a

[25] See for instance Murto and Tervio (2014), Décamps, Gryglewicz, Morellec and Villeneuve (2017), Bolton, Wang and Yang (2019), Babenko and Tserlukevich (2021) for corporate models with permanent shocks.

[26] We thank the referee for this suggestion.

[27] See DMRV (2011) for a complete mathematical analysis in a one-dimensional model. See Reppen, Rochet and Soner (2020) for numerical approximations in a two-dimensional model.

[28] See for instance Décamps and Villeneuve (2007), Bolton Chen and Wang (2011), Bolton, Wang and Yang (2019), Babenko and Tserlukevich (2021).

sudden opportunity that requires cash.[26] Let us assume for instance that, at a random time cash constrained shareholders have the possibility to secure the firm's project by paying I and getting a positive non-random payment $\theta \leq \frac{\mu}{r}$. In this extended setting, the dynamics of the cash reserves follow Equation (6) and the cash policy aims to 1) delay liquidation, 2) learn about profitability, 3) seize the opportunity to secure the project. Intuitively, in the first best case where shareholders are deep pocket the firm finds itself in two different regions when the opportunity arises: 1) profitability prospects are high so that shareholders are confident about the actual profitability and seizing the opportunity to secure the project is not optimal whatever the level of cash reserves, 2) profitability prospects are low and securing the project is optimal. A financially constrained firm may not be able to secure its project because of lack of liquidity. A computation shows that if there is no cost of holding cash, a payout policy that maintains the level of cash reserves on a critical curve is optimal and yields the first best value of the firm. This finding generalizes a result of Gryglewicz (2011) to the case where an investment opportunity is available. Interestingly, an analogous property holds also in the two-dimensional model with complete information of Bolton, Wang and Yang (2019) where the profitability process follows a Geometric Brownian Motion and where there is no additional brownian shock in the dynamics of the cumulative cash flow process (with our notations, $\sigma = 0$). The full analysis with cost of holding cash requires further work. It is worth noting that, existing models focus either on investment and learning (see Andrei, Mann, Moyen (2019) for a recent contribution) either on investment and financial constraints.The above preliminary thoughts suggest that our model can serve as a workhorse model to study corporate investment and learning in a financially constrained framework.

Finally, as explained in the introduction, our model addresses the case of all-equity young firms that have little collateral to offer. Well-established firms with more elaborated financial structure face also corporate cash management issues, and learn about their profitability when they launch new projects or engage in major restructurings. Clearly, future learning models should integrate into the analysis a wider range of financial tools, especially debt issuance and the use of credit lines. Other learning issues arise in corporate finance. For instance, some studies focus on learning about the state of the economy, or learning about a rival's charateristics, nevertheless avoiding cash management considerations.[29] We lack of studies that integrate these learning issues in a setting of constrained financing. These and related questions must await for future work.

[29]See for instance, Grenadier and Malenko (2010), Décamps and Mariotti (2004).

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8 Appendix

8.1 First-best benchmark

Proposition 3.1 relies on the following Lemma.

Lemma 8.1 *The value of the firm V^* can be written in the form*

$$V^*(x,y) \;=\; x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\left(Y_s - rX_s\right)\, ds\right] \leq x + \frac{\mu}{r}. \tag{32}$$

Proof of Lemma 8.1. For all $t > 0$ and all admissible controls I and D, we have

$$e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} = x + \int_0^{t\wedge\tau_0} e^{-rs}dX_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\, ds.$$

Because X_t is nonnegative for $t \leq \tau_0$ and Y is bounded above by μ, we get

$$\begin{aligned} e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} + \int_0^{t\wedge\tau_0} e^{-rs}(dD_s - dI_s) &= x + \int_0^{t\wedge\tau_0} e^{-rs}dR_s - r\int_0^{t\wedge\tau_0} e^{-rs}X_s\, ds \quad (33)\\ &\leq x + \frac{\mu}{r} + \sigma\int_0^{t\wedge\tau_0} e^{-rs}dB_s. \end{aligned}$$

Applying the optional sampling theorem to the uniformly $\mathcal{F}^R$-martingale $\left(\int_0^t e^{-rs}dB_s\right)_{t\geq 0}$ and taking expectations lead to

$$\mathbb{E}\left[e^{-r(t\wedge\tau_0)}X_{t\wedge\tau_0} + \int_0^{t\wedge\tau_0} e^{-rs}(dD_s - dI_s)\right] \leq x + \frac{\mu}{r}.$$

Letting t goes to $+\infty$, we obtain for all $(x,y) \in [0,\infty)\times(-\mu,\mu)$

$$\mathcal{V}(x,y;I,D) \leq x + \frac{\mu}{r} < \infty.$$

Thus, the value function V^* is finite. We obtain (32) from equations (3) and (33). $\square$

Proof of Proposition 3.1. Let us define

$$\Gamma(y) = \sup_{\tau\in\mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs}Y_s ds\right].$$

Standard results in optimal stopping theory[30] yield that the optimal value function Γ is C^1 on $(-\mu,\mu)$ and that a threshold strategy $\hat\tau = \inf\{t \geq 0\,|\, Y_t = \hat y\}$ is optimal. The value function Γ and the threshold $\hat y$ can be written in terms of the free boundary problem: Find $\Gamma \in C^1((-\mu,\mu))$ and $\hat y \in (-\mu,\mu)$ such that,

$$\begin{cases} \frac{1}{2\sigma^2}(y+\mu)^2(\mu-y)^2\Gamma''(y) - r\Gamma(y) + y = 0, \; y \geq \hat y,\\ \Gamma(\hat y) = 0, \;\; \Gamma'(\hat y) = 0. \end{cases}$$

[30]See for instance Peskir and Shiryaev (2006).

Standard computations yield

$$\begin{cases} \hat{V}(x,y) = x, \quad -\mu < y \leq \hat{y}, \\ \hat{V}(x,y) = x + \frac{y}{r} - \frac{h_1(y)}{h_1(\hat{y})}\frac{\hat{y}}{r} \quad \hat{y} \leq y < \mu, \end{cases} \tag{34}$$

where

$$h_1(y) = (\mu+y)^{\gamma}(\mu-y)^{1-\gamma}, \;\; \hat{y} = \frac{-\mu}{1-2\gamma} < 0, \tag{35}$$

$$\text{and where } \gamma \text{ is the negative root of the equation } x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0. \tag{36}$$

Noting that $\hat{y} < 0$ and that h_1 is convex, we deduce from (34), that $\hat{V}(x,y)$ is increasing and convex in y.

Finally, we show that the function V^* is bounded above by $\hat{V}$. Because the cash reserves are positive for all admissible controls, it follows from (32) that,

$$x + \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right]$$

is an upper bound for V^*. From Equation (3), any admissible control (I,D) acts on $\mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right]$ by modifying only the $\mathcal{F}^R$-stopping time τ_0. Thus,

$$\sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} Y_s \, ds\right] \leq \sup_{\tau\in\mathcal{T}^R} \mathbb{E}\left[\int_0^{\tau} e^{-rs} Y_s ds\right],$$

which yields the desired inequality. □

8.2 Complete information benchmark.

In this section, we consider that the firm's profitability is known and is equal to μ. We develop the mathematical formulation of Proposition 3.2. This formulation yields useful formulae for the proof of Theorem 4.1.

8.2.1 No equity issuance

We start the analysis with the case where security issuances are not allowed. The dynamics of the cash reserves satisfy

$$dX_t = \mu \, dt + \sigma dB_t - dD_t,$$

and the shareholders' problem writes

$$\overline{V}_{\mu}(x) = \sup_{D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs} dD_s\right], \tag{37}$$

where $\tau_0 = \inf\{t \geq 0 \,|\, X_t = 0\}$. The following result is due to Jeanblanc and Shiryaev (1995).

Proposition 8.2 *The value function $\overline{V}_\mu$ of problem (37) is concave, twice continuously differentiable and it satisfies the following HJB equation on* $(0, +\infty)$*:*

$$\max\left\{\frac{\sigma^2}{2}\overline{V}''_\mu + \mu\overline{V}'_\mu - r\overline{V}_\mu, 1 - \overline{V}'_\mu\right\} = 0.$$

Moreover, we have

$$\begin{cases} \overline{V}_\mu(x) = \dfrac{e^{-\beta\gamma x} - e^{\beta(\gamma-1)x}}{-\beta\gamma e^{-\beta\gamma\overline{x}_\mu} + (1-\gamma)\beta e^{\beta(\gamma-1)\overline{x}_\mu}} & 0 \le x \le \overline{x}_\mu, \\ \overline{V}_\mu(x) = x - \overline{x}_\mu + \frac{\mu}{r}, \quad x \ge \overline{x}_\mu, \end{cases} \tag{38}$$

with

$$\overline{x}_\mu = \frac{1}{\beta(1-2\gamma)}\ln(\frac{1-\gamma}{\gamma})^2 = 2\frac{\hat{y}}{\mu}\phi(\hat{y}), \tag{39}$$

where (35) and (36) define $\hat{y}$ and γ and where $\beta = \frac{2\mu}{\sigma^2}$. Any excess of cash over the dividend boundary $\overline{x}_\mu$ is paid out to shareholders, such that the cash reserve process is reflected back each time it reaches $\overline{x}_\mu$. The process $D = \{D_t; t \ge 0\}$ with

$$D_t = (x - \overline{x}_\mu)^+ \mathbb{1}_{t=0} + L_t^{\overline{x}_\mu} \mathbb{1}_{t>0} \tag{40}$$

is the optimal dividend payment process. In equation (40), $L^{\overline{x}_\mu}$ denotes the so-called local time process solution to the Skohorod problem[31] *at $\overline{x}_\mu$ for the drifted Brownian motion $\mu t + B_t$.*

8.2.2 Equity issuance

When security issuances are allowed at a proportional issuance cost $p > 1$, the dynamics of the cash reserves satisfy

$$dX_t = \mu\, dt + \sigma dB_t - dD_t + \frac{dI_t}{p},$$

and shareholders' problem writes

$$V_\mu(x) = \sup_{I,D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}(dD_s - dI_s)\right], \tag{41}$$

where $\tau_0 = \inf\{t \ge 0 \,|\, X_t = 0\}$. The following proposition is due to Lokka and Zervos (2008) and provides a rigorous formulation of Proposition 3.2 in the main text.

Proposition 8.3 *The value function defined in (41) is concave, twice continously differentiable and it satisfies the following HJB equation on* $(0, +\infty)$*:*

$$\max\{\frac{\sigma^2}{2}V''_\mu + \mu V'_\mu - rV_\mu, 1 - V'_\mu, V'_\mu - p\} = 0.$$

Moreover, we have

- *If $p \ge \overline{V}'_\mu(0)$ then, $V_\mu(x) = \overline{V}_\mu(x)$ for all $x \ge 0$.*

[31] See Karatzas and Shreve (1991) page 210.

- *If* $p < \overline{V}'_\mu(0)$ *then,*

$$\begin{cases} V_\mu(x) = \dfrac{1-\gamma}{\beta\gamma}\dfrac{\hat{y}}{\mu}e^{-\beta\gamma(x-x_\mu)} + \dfrac{\gamma}{\beta(\gamma-1)}\dfrac{\hat{y}}{\mu}e^{\beta(\gamma-1)(x-x_\mu)} \quad 0 \le x \le x_\mu, \\ V_\mu(x) = x - x_\mu + \frac{\mu}{r}, \quad x \ge x_\mu, \end{cases} \tag{42}$$

where x_μ *is defined as the unique solution to the equation*

$$p = -\frac{\hat{y}}{\mu}(1-\gamma)e^{\gamma\beta x_\mu} + \frac{\hat{y}}{\mu}\gamma e^{(1-\gamma)\beta x_\mu}. \tag{43}$$

Any excess of cash over the dividend boundary x_μ *is paid out to shareholders, so that the cash reserve process is reflected back each time it reaches* x_μ*. There is equity issuance whenever the firm runs out of cash, so that the cash reserve process is reflected back each time it reaches* 0*. The processes* $D = \{D_t; t \ge 0\}$ *and* $I = \{I_t; t \ge 0\}$ *with*

$$D_t = (x - x_\mu)^+ \mathbb{1}_{t=0} + L_t^{x_\mu}\mathbb{1}_{t>0} \ \text{ and } \ I_t = L_t^0 \mathbb{1}_{t>0} \tag{44}$$

are the optimal dividend payment and equity issuance processes. In equation (44), L^{x_μ} *and* L^0 *denote the solution to the Skohorod problem at* x_μ *and at* 0 *for the drifted Brownian motion* $\mu t + B_t$*.*

Hereafter, we will note $\overline{p} = \overline{V}'_\mu(0)$. The thresholds $\overline{x}_\mu$ and x_μ defined in (39) and (43) satisfy $\overline{x}_\mu > x_\mu$. Moreover, equation (38) yields that $\overline{p} = \frac{1-\gamma}{-\gamma}e^{\gamma\beta\overline{x}_\mu}$. We deduce that for $1 < p < \overline{p}$, we have

$$p < \frac{1-\gamma}{-\gamma}e^{\beta\gamma x_\mu}. \tag{45}$$

We will use later this inequality.

8.3 Model Solution

We devote this section to the proof of Theorem 4.1. As for the complete information benchmark, it is useful to start the analysis under the assumption that the shareholders are not allowed to issue new shares.

8.3.1 No equity issuance

Thus, in this subsection we solve the problem

$$\overline{V}(x,y) = \sup_{D\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}dD_t\right], \tag{46}$$

where $\tau_0 = \inf\{t \ge 0 \,|\, X_t = 0\}$ with $X_t = \phi(Y_t) - \phi(y) + x - D_t$.

As a preliminary but essential step, we establish a standard verification Lemma that specifies conditions under which a function V defined on $[0,\infty) \times (-\mu,\mu)$ is a majorant of the value function $\overline{V}$ of the problem (46).

Lemma 8.4 *(Verification Lemma) Assume there exists a function* V *defined on* $[0,\infty) \times (-\mu,\mu)$ *that satisfies*

1. *V is twice differentiable,*

2. *V has bounded first derivatives,*

3. *$V(0,y)=0$ for all $y\in(-\mu,\mu)$ and*

$$\max(\mathcal{L}V - rV, 1-V_x) \leq 0 \text{ on } [0,\infty)\times(-\mu,\mu),$$

then V is a majorant of $\overline{V}$.

Proof of Lemma 8.4. See the online appendix. □

Second, we explicitly build such a majorant. To this end, we prove that the following variational problem

$$\begin{aligned}
\mathcal{L}V(x,y) - rV(x,y) &= 0 \quad \text{on the domain } \{(x,y),\ 0<x<b(y),\ -\mu<y<\mu\}, && (47)\\
V(0,y) &= 0 \quad \forall y\in(-\mu,\mu), && (48)\\
V_x(x,y) &= 1,\ \text{for } x\geq b(y), && (49)\\
V_{xy}(b(y),y) &= 0, && (50)\\
\lim_{y\longrightarrow\mu} V(x,y) &= \overline{V}_\mu(x)\ \ \forall x\geq 0, && (51)
\end{aligned}$$

has a unique solution (V,b) (see Proposition 8.6), such that V satisfies Lemma 8.4 (see Proposition 8.7) and thus dominates $\overline{V}$. Finally, we show in Proposition 8.8 that V can be reached by an admissible policy and thus coincides with the solution $\overline{V}$ to the problem (46). This last step also provides the optimal dividend policy and concludes the study of problem (46).

We start with a technical lemma.

Lemma 8.5 *The ordinary differential equation*

$$\begin{aligned}
g'(y) &= f(g(y),y), && (52)\\
g(\mu) &= \overline{x}_\mu, && (53)
\end{aligned}$$

with

$$f(x,y) = \frac{\sigma^2}{\mu^2-y^2}\frac{y\hat{y} + \left(\mu - r\sigma^2\left(\frac{\hat{y}}{\mu}\right)^2\frac{1}{\mu}\right)\phi^{-1}\left(-\frac{\mu}{\hat{y}}\,x\right)}{y\hat{y}\ +\ \mu\phi^{-1}\left(-\frac{\mu}{\hat{y}}\,x\right)} \qquad (54)$$

defined on the domain $\{(x,y)\in[0,\infty)\times(-\mu,\mu)\,|\,x>\frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu})\}$ has a unique solution. The solution g is C^1 and increasing over $[\overline{y}^*,\mu]$ where the threshold $\overline{y}^*\equiv g^{-1}(0)$ is well defined and strictly larger than $\hat{y}$. Moreover, if we define $\overline{b}=\max(g,0)$, then $\overline{k}=(\phi-\overline{b})^{-1}:[\phi(\overline{y}^*),\infty)\longrightarrow[\overline{y}^*,\mu)$ is a well defined C^1 increasing function. The function $\overline{k}$ is the unique solution to the ordinary differential equation

$$\begin{aligned}
\overline{k}'(z) &= \Theta(z,\overline{k}(z)), && (55)\\
\lim_{z\longrightarrow\infty}\phi(\overline{k}(z)) - z &= \overline{x}_\mu, && (56)
\end{aligned}$$

with

$$\Theta(z,y) = \frac{\mu^3}{\hat{y}r\sigma^2}\frac{\mu^2-y^2}{\sigma^2}\frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(y)-z)\right)\mu - y\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(y)-z)\right)} \tag{57}$$

defined on the domain $\{(z,y)\in\mathbb{R}\times(-\mu,\mu)\,|\,\phi(y)-z>\max(0,\frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}))\}$.

Proof of Lemma 8.5. See the online appendix □

We are now in a position to solve the system (47)-(51).

Proposition 8.6 *Let us consider the functions* $\overline{b}$ *and* $\overline{k}$ *defined in Lemma 8.5 and the function* $(x,y)\longrightarrow V(x,y)$ *defined on* $[0,\infty)\times(-\mu,\mu)$ *by the relations*

$$\begin{cases} V(0,y)=0, \quad \textit{for } y\in(-\mu,\mu),\\ V(x,y)=A\left(\phi(y)-x\right)\left(h_1(y)-e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y)-x)}h_2(y)\right), \quad \textit{for } 0\le x\le\overline{b}(y), y\in(-\mu,\mu),\\ V(x,y)=x-\overline{b}(y)+V(\overline{b}(y),y), \quad \textit{for } x\ge\overline{b}(y), y\in(-\mu,\mu), \end{cases} \tag{58}$$

where

$$A(z) = \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(\overline{k}(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}-h_2'(\overline{k}(z))\right). \tag{59}$$

Then, the couple $(V,\overline{b})$ *is the unique solution to the system (47)-(51). Furthermore, the function* $\overline{b}:[\overline{y}^*,\mu]\longrightarrow[0,\overline{x}_\mu]$ *is* C^1 *and increasing.*

Proof of Proposition 8.6. Having in mind the change of variable (18), we are looking for a smooth function U defined on $[0,\infty)\times(-\mu,\mu)$ and a C^1 function $k:\mathbb{R}\longrightarrow(-\mu,\mu)$ that solve the variational system

$$\begin{aligned} \frac{1}{2\sigma^2}(\mu^2-y^2)^2U_{yy}(z,y)-rU(z,y) &= 0 \text{ on } \{(z,y),\ z\in\mathbb{R},\ \psi(z)<y<k(z)\}, && (60)\\ U(z,\psi(z)) &= 0, \text{ for } z\in\mathbb{R}, && (61)\\ U_z(z,y) &= -1, \text{ for } k(z)\le y, && (62)\\ U_{xy}(z,k(z)) &= 0, && (63)\\ \lim_{z\longrightarrow\infty}U(z,\psi(x+z)) &= \overline{V}_\mu(x). && (64) \end{aligned}$$

First, we establish a set of necessary conditions for the existence of such a pair (U,k) by observing that any solution to the o.d.e. (60) can be written in the form

$$U(z,y)=A(z)h_1(y)+B(z)h_2(y), \tag{65}$$

where $h_1(y)=(y+\mu)^\gamma(\mu-y)^{1-\gamma}$ and $h_2(y)=(y+\mu)^{1-\gamma}(\mu-y)^\gamma$. Using (65), we obtain from (62) and (63) that

$$A'(z)=h_2'(k(z))\frac{\hat{y}}{2\mu^2}, \text{ and } B'(z)=-h_1'(k(z))\frac{\hat{y}}{2\mu^2}. \tag{66}$$

Using again (65), we rewrite (61) in the form

$$A(z) = -B(z)\frac{h_2(\psi(z))}{h_1(\psi(z))} = -B(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}. \tag{67}$$

Taking the derivative of (67), we obtain

$$A'(z) = -B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z} + B(z)\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}$$

which yields using again (67),

$$A(z) = -\frac{\sigma^2\hat{y}}{2\mu^2}\left(A'(z) + B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}\right). \tag{68}$$

Using (65) and (67), and plugging (66) into (68) yield that,

$$U(z,y) = A(z)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(z)}h_2(y)\right), \tag{69}$$

where

$$A(z) \;=\; \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z} - h_2'(k(z))\right). \tag{70}$$

Taking the derivative of (68) and using (66), we obtain that

$$k'(z) = \frac{2\mu^2}{\sigma^2\hat{y}}\frac{h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z))}{h_1''(k(z))h_2(\psi(z)) - h_2''(k(z))h_1(\psi(z))}. \tag{71}$$

A computation yields that

$$k'(z) = \frac{\mu^3}{\hat{y}r\sigma^2}\frac{\mu^2 - k(z)^2}{\sigma^2}\frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(k(z)) - z)\right)\mu - k(z)\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(k(z)) - z)\right)}, \tag{72}$$

which is positive on the domain $\phi(k(z)) - z > \max(0, \frac{\hat{y}}{\mu}\phi(\frac{\hat{y}}{\mu}k(z)))$.

Thus, (69), (70) and (72) is a set of necessary conditions for the existence of a smooth solution (U,k) to (60), (61), (62), (63). It remains to find a necessary condition for a solution to satisfy (64). Below we prove that U satisfies (64) if and only if $\lim_{z\longrightarrow\infty}\phi(k(z)) - z = \overline{x}_\mu$. From Lemma 8.5, it will imply that $k = \overline{k}$. To do this, we use (69) and (70) to obtain

$$U(z,\psi(x+z)) = \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2 f(x)\Delta(z)\frac{1+e^{-\beta z}}{1+e^{-\beta(x+z)}}, \tag{73}$$

with

$$f(x) = e^{(\gamma-1)\beta x}(1 - e^{(1-2\gamma)\beta x}),\ \Delta(z) = h_1'(k(z))h_2(\psi(z)) - h_2'(k(z))h_1(\psi(z)).$$

Observing that $k(z) = \psi((\phi(k(z)) - z) + z)$, a computation yields the following asymptotics

$$h_1'(k(z))h_2(\psi(z)) \sim_{z=\infty} 2\mu(\gamma-1)e^{\gamma\beta(\phi(k(z))-z)}, \quad h_2'(k(z))h_1(\psi(z)) \sim_{z=\infty} -2\mu\gamma e^{(1-\gamma)\beta(\phi(k(z))-z)},$$

which imply

$$\lim_{z\longrightarrow\infty} \Delta(z) = \lim_{z\longrightarrow\infty} 2\mu\left((\gamma-1)e^{\gamma\beta(\phi(k(z))-z)} - \gamma e^{(1-\gamma)\beta(\phi(k(z))-z)}\right). \tag{74}$$

Using (38), we observe that

$$\overline{V}_\mu(x) = f(x)\frac{e^{\beta\gamma\overline{x}_\mu}}{\beta(\gamma + (\gamma-1)e^{(2\gamma-1)\beta\overline{x}_\mu})}. \tag{75}$$

It then follows from (73), (74), (75) that $\lim_{z\longrightarrow\infty} U(z, \psi(x+z)) = \overline{V}_\mu(x)$ is equivalent to $\lim_{z\longrightarrow\infty} \phi(k(z)) - z = \overline{x}_\mu$. Thus, a smooth solution (U, k) to (60)-(64), if it exists, must satisfy (69), (70) where $k = \overline{k}$ is uniquely defined in Lemma 8.5. Conversely, a direct computation shows that the function defined by (69), (70) with $k = \overline{k}$ is a smooth solution to (60)-(64). Finally, posing $z = \phi(y) - x$ in (69) and (70) leads to (58) and (59) where $\overline{b}$ is uniquely defined in Lemma 8.5. Thus, the couple $(V, \overline{b})$ defined in Proposition 8.6 is the unique solution to the system (47)-(51). Observe that the uniqueness of the function V comes from the uniqueness of the function $\overline{b}$ which follows from condition (64). □

To prove that $V = \overline{V}$, we proceed in two steps. First, we show in Proposition 8.7 that the function V solution to (60)-(64) satisfies the assumptions of the verification Lemma 8.4, which implies that $\overline{V} \leq V$. Second, we construct an admissible policy for problem (46), the value of which coincides with V. This latter result implies that $V \leq \overline{V}$.

Proposition 8.7 *The function V defined in Proposition 8.6 satisfies the assumptions of Lemma 8.4.*

Proof of Proposition 8.7. It is clear from (58) that V is twice continuously differentiable on any open set in $(0,\infty) \times (-\mu,\mu)$ away from the set $\{(x,y),\ x = \overline{b}(y)\}$. By construction V_x and V_{xx} are continuous across the boundary $\overline{b}$. Therefore, to prove that V is twice differentiable on $(0,\infty) \times (-\mu,\mu)$, we only have to show that the functions V_y and V_{yy} are continuous across the boundary $\overline{b}$, that is

$$\lim_{x\longrightarrow\overline{b}(y)^-} V_y(x,y) = -\overline{b}'(y) + \nu'(y), \lim_{x\longrightarrow\overline{b}(y)^-} V_{yy}(x,y) = -\overline{b}''(y) + \nu''(y), \tag{76}$$

where the function ν is defined on $(-\mu,\mu)$ by the relation

$$\nu(y) = A(\phi(y) - \overline{b}(y))\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y)-\overline{b}(y))}h_2(y)\right).$$

Let us define

$$H(y) \equiv e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y)-\overline{b}(y))} = \frac{h_1(\psi(\phi(y)-\overline{b}(y))}{h_2(\psi(\phi(y)-\overline{b}(y))}.$$

A computation yields

$$H'(y) = \frac{2\mu^2}{\sigma^2 \hat{y}}(\phi'(y) - \overline{b}'(y))H(y). \tag{77}$$

Remembering the relations (66) and (68) and the definition of $\overline{k}$, we observe that

$$A(z) = -\frac{\sigma^2 \hat{y}}{2\mu^2}\left(A'(\phi(y) - \overline{b}(y)) + B'(\phi(y) - \overline{b}(y))\frac{1}{H(y)}\right).$$

We are in a position to compute the derivative of ν. We have,

$$\begin{aligned}
\nu'(y) &= A(\phi(y) - \overline{b}(y))(h_1'(y) - H(y)h_2'(y)) \\
&+ (\phi'(y) - \overline{b}'(y))A'(\phi(y) - \overline{b}(y))(h_1(y) - H(y)h_2(y)) - A(\phi(y) - \overline{b}(y))H'(y)h_2(y).
\end{aligned}$$

Using the relations (77), (68), (66) and the definition of $\overline{k}$, the second term of the right-hand side is equal to

$$(\phi'(y) - \overline{b}'(y))\frac{\hat{y}}{2\mu^2}(h_2'(y)h_1(y) - h_1'(y)h_2(y)).$$

We note that the change of variable $V(x,y) = U(\phi(y) - x, y)$ leads to the relations

$$V_y(x,y) = \phi'(y)U_z(\phi(y) - x, y) + U_y(\phi(y) - x, y), \tag{78}$$

$$\begin{aligned}
V_{yy}(x,y) = \phi''(y)U_z(\phi(y) - x, y) + \phi'(y)^2 U_{zz}(\phi(y) - x, y) \\
+2\phi'(y)U_{zy}(\phi(y) - x, y) + U_{yy}(\phi(y) - x, y).
\end{aligned} \tag{79}$$

As a consequence,

$$\begin{aligned}
\nu'(y) &= (\phi'(y) - \overline{b}'(y))\frac{\hat{y}}{2\mu^2}((h_2'(y)h_1(y) - h_1'(y)h_2(y)) \\
&\quad + A(\phi(y) - \overline{b}(y))(h_1'(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y) - \overline{b}(y))}h_2'(y)) \\
&= -\phi'(y) + \overline{b}'(y) + U_y(\phi(y) - \overline{b}(y), y) = \overline{b}'(y) + \lim_{x \longrightarrow \overline{b}(y)^-} V_y(x,y),
\end{aligned}$$

where the first equality comes from (66) and the last equality comes from (78) and from the relation $U_z(\phi(y) - \overline{b}(y), y) = -V_x(\overline{b}(y), y) = -1$. Thus (76) is satisfied. Moreover,

$$\begin{aligned}
\nu''(y) &= -\phi''(y) + \overline{b}''(y) + A((\phi(y) - \overline{b}(y))h_1''(y) + B((\phi(y) - \overline{b}(y))h_2''(y) \\
&\quad + (\phi'(y) - \overline{b}'(y))A'(\phi(y) - \overline{b}(y))(h_1'(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y) - \overline{b}(y))}h_2'(y)) \\
&= -\phi''(y) + \overline{b}''(y) + \frac{2r\sigma^2}{(\mu^2 - y^2)^2}U((\phi(y) - \overline{b}(y), y) \\
&\quad + (\phi'(y) - \overline{b}'(y))U_{zy}((\phi(y) - \overline{b}(y), y) \\
&= -\phi''(y) + \overline{b}''(y) + \frac{2r\sigma^2}{(\mu^2 - y^2)^2}U((\phi(y) - \overline{b}(y), y) = \overline{b}''(y) + \lim_{x \longrightarrow \overline{b}(y)^-} V_{yy}(x,y),
\end{aligned}$$

where the last equality comes from (79) and from the relations $U_{zz}(\phi(y) - \overline{b}(y), y) = U_{zy}(\phi(y) - \overline{b}(y), y) = 0$, $U_z(\phi(y) - \overline{b}(y), y) = -1$ and $U_{yy}(\phi(y) - \overline{b}(y), y) = \frac{2r\sigma^2}{(\mu^2 - y^2)^2}U((\phi(y) - \overline{b}(y), y)$. Therefore, V is twice differentiable on $(0, \infty) \times (-\mu, \mu)$.

We show below that the function V has bounded first derivatives V_x and V_y. This amounts to show that $\lim_{x\longrightarrow\infty} V_x(x,y) < \infty$ and $\lim_{y\longrightarrow\mu} V_y(x,y) < \infty$. We write the change of variable $x = \phi(y) - z$ in the form $y = \psi(x+z)$ and define

$$W(x,z) \equiv V(x,\psi(x+z)) = U(z,\psi(x+z)).$$

We have

$$\begin{aligned} V_x(x,y) &= W_x(x,\phi(y)-x) - W_z(x,\phi(y)-x), \\ V_y(x,y) &= \phi'(y)W_z(x,\phi(y)-x) = \frac{\sigma^2}{\mu^2-y^2}W_z(x,\phi(y)-x). \end{aligned}$$

We then deduce that V_x is bounded if and only if

$$\lim_{z\longrightarrow\infty} W_x(x,z) - W_z(x,z) < \infty. \tag{80}$$

A computation shows that V_y is bounded if and only if $\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)}W_z(x,z) < \infty$, or, equivalently if and only if

$$\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)}\Delta'(z) < \infty, \tag{81}$$

where the latter expression follows from a computation that uses (73). Using (71) we obtain that (81) is equivalent to

$$\begin{aligned} &\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)}\frac{2\mu^2}{\sigma^2\hat{y}}(h_1'(\overline{k}(z))h_2(\psi(z)) + h_2'(\overline{k}(z))h_1(\psi(z))) \\ &\quad + \frac{\psi'(z)}{\psi'(x+z)}(h_1'(\overline{k}(z))h_2'(\psi(z)) - h_2'(\overline{k}(z))h_1'(\psi(z))) < \infty. \end{aligned} \tag{82}$$

Recalling that $\lim_{z\longrightarrow\infty} \phi(\overline{k}(z)) - z = \overline{x}_\mu$ and observing that $\overline{k}(z) = \psi((\phi(\overline{k}(z)) - z)$, computations lead to the relations

$$\begin{aligned} \frac{1}{\psi'(x+z)}\frac{2\mu^2}{\sigma^2\hat{y}}h_1'(\overline{k}(z))h_2(\psi(z)) &\sim_{z=\infty} \frac{\mu}{\hat{y}}(\gamma-1)e^{\beta x}e^{\gamma\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{1}{\psi'(x+z)}\frac{2\mu^2}{\sigma^2\hat{y}}h_2'(\overline{k}(z))h_1(\psi(z)) &\sim_{z=\infty} -\frac{\mu}{\hat{y}}\gamma e^{\beta x}e^{(1-\gamma)\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x+z)}h_1'(\overline{k}(z))h_2'(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma-1)e^{\beta x}e^{\gamma\beta\overline{x}_\mu}e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x+z)}h_2'(\overline{k}(z))h_1'(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma-1)e^{\beta x}e^{(1-\gamma)\beta\overline{x}_\mu}e^{\beta z}. \end{aligned}$$

Aggregating these relations in (82), we obtain that

$$\lim_{z\longrightarrow\infty} \frac{1}{\psi'(x+z)}W_z(x,z) = \lim_{z\longrightarrow\infty} e^{\beta x}e^{\beta z}((\gamma-1)^2e^{\gamma\beta\overline{x}_\mu} - \gamma^2e^{(1-\gamma)\beta\overline{x}_\mu}) = 0, \tag{83}$$

where the last equality follows from (39). Thus, V_y is bounded. We deduce from (83) that $\lim_{z \longrightarrow \infty} W_z(x,z) = 0$ and from (73) that

$$W_x(x,z) \sim_{z=\infty} \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2 f'(x)\Delta(z).$$

Since $\lim_{z \longrightarrow \infty} \phi(\overline{k}(z)) - z = \overline{x}_\mu$, we obtain from (74) that (80) is satisfied. Thus V_x is bounded. Finally, we prove below that

$$\max(\mathcal{L}V - rV, 1 - V_x) \leq 0 \text{ on } [0,\infty) \times (-\mu, \mu).$$

Note that, by construction, $V(0,y) = 0$ for all $y \in (-\mu,\mu)$ and that $\mathcal{L}V - rV = 0$ on the set $\{x \leq \overline{b}(y)\}$. We first show that the mapping $x \longrightarrow V(x,y)$ is concave on the set $\{x < \overline{b}(y)\}$. The change of variable (18) and relation (60) yield that, on the set $\{x \leq \overline{b}(y)\}$,

$$V(x,y) = A(\phi(y) - x)h_1(y) + B(\phi(y) - x)h_2(y).$$

Using (66), we obtain that

$$V_{xx}(x,y) = \overline{k}'(\phi(y) - x)\frac{\hat{y}}{2\mu^2}(h_2''(\overline{k}(\phi(y) - x)h_1(y) - h_1''(\overline{k}(\phi(y) - x)h_2(y)). \tag{84}$$

The right hand side of (84) has the same sign than

$$\overline{k}'(\phi(y) - x)\frac{\hat{y}}{2\mu^2}(h_2(\overline{k}(\phi(y) - x)h_1(y) - h_1(\overline{k}(\phi(y) - x)h_2(y)). \tag{85}$$

Since h_1 is positive decreasing and h_2 is positive increasing, the function $(h_2(\overline{k}(\phi(y) - x)h_1(y) - h_1(\overline{k}(\phi(y) - x)h_2(y))$ is positive if and only if $\overline{k}(\phi(y) - x) > y$. Since $\overline{k}$ is increasing, this latter inequality is equivalent to $\phi(y) - x > \overline{k}^{-1}(y)$, that is $x < \overline{b}(y)$. Thus, (85) is negative since $\hat{y} < 0$. Therefore, the mapping $x \longrightarrow V(x,y)$ is concave on the set $\{x < \overline{b}(y)\}$. Because $V_x(x,y) = 1$ for all $x \geq \overline{b}(y)$, we conclude that $x \longrightarrow V(x,y)$ is concave over $[0,\infty)$, and in turn that $V_x \geq 1$ on $[0,\infty)$.

It remains to show that $\mathcal{L}V - rV < 0$ on the set $\{x > \overline{b}(y)\}$. On the set $\{x > \overline{b}(y)\}$, we have that $V(x,y) = x - \overline{b}(y) + \nu(y)$ and $V_{xy}(\overline{b}(y),y) = 0$. We deduce the equalities $V_y(x,y) = V_y(\overline{b}(y),y)$ and $V_{yy}(x,y) = V_{yy}(\overline{b}(y),y)$. Therefore, using the fact that V is twice differentiable across $\overline{b}$, we obtain that, on the set $\{x > \overline{b}(y)\}$,

$$(\mathcal{L}V - rV)(x,y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy}(\overline{b}(y),y) + y - rV(x,y)) = -r(x - \overline{b}(y)) < 0.$$

The proof of Proposition 8.7 is complete and thus $\overline{V} \leq V$. □

Finally, we show that the solution V can be reached by an admissible policy. Our guess is that the optimal cash reserve process is reflected along the free boundary function $\overline{b}$ on a horizontal direction in the (x,y)-plane. We formalize this using a 2-dimensional version to Skorohod's lemma established by Burdzy and Toby (1995). Specifically, there exists a unique continuous process $\{L = (L_t)_t;\ t \geq 0\}$ defined on $(\Omega, \mathcal{F}^R, \mathbb{P})$ such that, for $\mathbb{P}$-*a.e.* $\omega \in \Omega$,

- $$(\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in [0, \overline{b}(y)] \times [\overline{y}^*, \mu), \quad \forall t \in [0, \tau_0], \tag{86}$$

where $\tau_0 = \inf\{t \geq 0 \,|\, \phi(Y_t) - \phi(y) + x - L_t = 0\}$,

- $$L_0(\omega) = 0, \text{ and } t \longrightarrow L_t(\omega) \text{ is nondecreasing on } \{t \geq 0 \,:\, \phi(Y_t) - \phi(y) + x - L_t = \overline{b}(Y_t)\}, \tag{87}$$

- $$t \longrightarrow L_t(\omega) \text{ is constant on } \{t \geq 0 \,:\, (\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in (0, \overline{b}(y)) \times (\overline{y}^*, \mu)\}. \tag{88}$$

Conditions (86)-(88) ensure that the policy L is admissible and that the process $\phi(Y_t) - \phi(y) + x - L_t$ is reflected in a horizontal direction whenever it hits $\overline{b}(Y_t)$.

Proposition 8.8 *The function V can be attained by an admissible policy and thus $V \leq \overline{V}$.*

Proof of Proposition 8.8 Let us consider the process $\overline{D} = \{\overline{D}_t; t \geq 0\}$ with

$$\overline{D}_t = \left((x - \overline{b}(y))^+ \mathbb{1}_{y \geq \overline{y}^*} \, + \, x \mathbb{1}_{y \leq \overline{y}^*}\right) \mathbb{1}_{t=0} \, + \, L_t \mathbb{1}_{t>0}, \tag{89}$$

where L is defined by (86)-(88) and let us consider the continuous process

$$X_t \equiv \phi(Y_t) - \phi(y) + x - \overline{D}_t.$$

A computation based on Itô's formula yields that, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E}\left[e^{-rt\wedge\tau_0} V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0})\right] &= V(x,y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs} V_x(X_s, Y_s)\, d\overline{D}_s\right] \\ &= V(x,y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\, d\overline{D}_s\right] \end{aligned} \tag{90}$$

where the second equality comes from (87) and (88) along with the fact that $V_x(\overline{b}(y), y) = 1$. Letting t go to ∞ in (90) yields

$$V(x,y) = \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\, d\overline{D}_s\right] \leq \overline{V}(x,y).$$

□

Thus, from Propositions 8.7 and 8.8, the function V defined in Proposition 8.6 coincides with the value function $\overline{V}$ of problem (46). Equation (89) provides the optimal dividend policy: The function $\overline{b}$ corresponds to the dividend boundary of the shareholders' problem (46). The optimal cash reserve process is reflected along the function $\overline{b}$ on a horizontal direction in the (x, y)-plane.

8.3.2 Equity issuance

We are now ready to prove Theorem 4.1 and to solve problem (8):

$$V^*(x,y) = \sup_{(I,D)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t - dI_t)\right].$$

We proceed as in the previous section where equity issuance was not allowed. The verification Lemma 8.4 has to be adapted in the following way.

Lemma 8.9 *(Verification Lemma) Assume there exists a function V defined on $[0,\infty)\times(-\mu,\mu)$ that satisfies*

1. *V is twice differentiable almost everywhere,*
2. *V has bounded first derivatives,*
3. $\max(-V(0,y), V_x(0,y) - p) = 0$ *for all $y \in (-\mu,\mu)$ and,*

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0 \text{ almost everywhere on } [0,\infty)\times(-\mu,\mu),$$

then $V \geq V^*$.

Proof of Lemma 8.9. See the online appendix.

We first assume that the proportional issuance costs p satisfies $p \geq \overline{p} = \overline{V}'_\mu(0)$. We prove that the firm value V^* solution to (8) coincides with $\overline{V}$ solution to (46), so that also the functions b^* and $\overline{b}$ coincide. This proves Theorem 4.1 when $p \geq \overline{p}$. The result is a direct consequence of the following Lemma.

Lemma 8.10 *Let us assume that $p \geq \overline{p}$, then $\overline{p} > \overline{V}_x(x,y)$ for all $(x,y) \in [0,\infty)\times \in (-\mu,\mu)$.*

Proof of Lemma 8.10. See the online appendix.

From Lemma 8.10, $\overline{V}$ satisfies the assumptions of Lemma 8.9. It follows that $V^*(x,y) \leq \overline{V}(x,y)$. On the other hand, considering the policies $I^* = 0$ and D^* defined in (89) lead to the inequality $V^*(x,y) \geq \overline{V}(x,y)$, thus the result.

The case $p \leq \overline{p}$ is much more involved. The analysis relies on the following technical Proposition.

Proposition 8.11 *The following holds.*

(i) Fix $z_i > \hat{z} = \phi(\hat{y})$, the relation

$$-h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p = 0 \tag{91}$$

uniquely defines over $[z_i, \infty)$ a continuously differentiable increasing function $k > \psi$ that satisfies

$$\lim_{z \longrightarrow \infty} \phi(k(z)) - z = x_\mu, \tag{92}$$

$$k(z) \underset{+\infty}{\sim} \psi(z + x_\mu). \tag{93}$$

(ii) The equation

$$\frac{p}{\psi'(z)} h_1(\psi(z)) + \int_z^\infty h_1'(k(u))\, du = 0 \tag{94}$$

has a unique solution $z_i^* \in (\hat{z}, \infty)$.

(iii) Let us denote $y_i^* = \psi(z_i^*)$ and let us consider the function k^* with

$$k^*(z) = k_1(z) 1\!\!1_{z \le z_i^*} + k_2(z) 1\!\!1_{z \le z_i^*}, \tag{95}$$

where (91) characterizes k_2, and where k_1 is the solution to the ordinary differential equation $k_1'(z) = \Theta(k_1(z), z)$ with terminal condition $k_1(z_i^*) = k_2(z_i^*)$ where Θ is defined by (57). Then, k^* is a well defined continuous increasing function over $(\hat{z}, \infty)$ and continuously differentiable over $\mathbb{R} \setminus \{z_i^*\}$.

Proof of Proposition 8.11. See the online appendix. □

We state now the main result of this section.

Proposition 8.12 *Let us consider the function k^* defined by (95) and the function b^* defined by the relation*

$$b^*(y) = \max(0, b_1(y)) 1\!\!1_{y \le y_i^*} + b_2(y) 1\!\!1_{y \ge y_i^*}, \tag{96}$$

with $b_2(y) = (\phi - k_2^{-1})(y)$ for any $y \ge y_i^*$ and, $b_1(y) = (\phi - k_1^{-1})(y)$ for any $y \le y_i^*$. The function b^* is well defined and positive on $[y^*, \mu)$ with $y^* = b^{-1}(0)$. It is differentiable on $(y^*, \mu) \setminus \{y^{i*}\}$ where $y^{i*} \equiv k_1(z_i^*) = k_2(z_i^*)$. Moreover, let us consider the function $(x, y) \longrightarrow V(x, y)$ defined on $\mathbb{R}^+ \times (-\mu, \mu)$ by the relations

- *For $y \in [y_i^*, \mu)$,*

$$\begin{cases} V(x,y) = A\,(\phi(y) - x)\, h_1(y) + B\,(\phi(y) - x)\, h_2(y), & \forall\, 0 \le x \le b^*(y), \\ V(x,y) = x - b^*(y) + V(b^*(y), y), & \forall\, x \ge b^*(y), \end{cases}$$

where for $z > z_i^*$:

$$\begin{cases} A(z) = A(z_i^*) + \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_2'(k^*(u))\, du, \\ B(z) = B(z_i^*) - \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_1'(k^*(u))\, du \end{cases} \tag{97}$$

and,

$$\begin{cases} A(z_i^*) = \frac{p}{\psi'(z_i^*)} \frac{\hat{y}}{2\mu^2} h_2(\psi(z_i^*)), \\ B(z_i^*) = -\frac{p}{\psi'(z_i^*)} \frac{\hat{y}}{2\mu^2} h_1(\psi(z_i^*)). \end{cases} \tag{98}$$

- *For $y \in (-\mu, y_i^*]$,*

$$\begin{cases} V(0,y) = 0, \quad \forall -\mu < y < \underline{y}, \\ V(x,y) = A\left(\phi(y) - x\right)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}(\phi(y)-x)}h_2(y)\right), \quad \forall\, 0 \le x \le b^*(y), \\ V(x,y) = x - b^*(y) + V(b^*(y),y), \quad \forall\, x \ge b^*(y) \end{cases}$$

where for $z \le z_i$,

$$A(z) = \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k^*(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z} - h_2'(k^*(z))\right). \tag{99}$$

Then, the triple (V, y_i^*, b^*) is the unique solution to the system (12)-(17).

Proof of Proposition 8.12. The proof follows the same route than the proof of Proposition 8.6. We first consider a solution (U, z_i, k) to the system

$$\begin{aligned}
\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z,y) - rU(z,y) &= 0 \quad \text{on } \{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}, && (100)\\
U(z,\psi(z)) &= 0 \quad \forall z \le z_i, && (101)\\
U_z(z,\psi(z)) &= -p \quad \forall z \ge z_i, && (102)\\
U_z(z,y) &= -1, \text{ for } k(z) \le y, && (103)\\
U_{xy}(z,k(z)) &= 0, && (104)\\
\lim_{z \longrightarrow \infty} U(z,\psi(x+z)) &= V_\mu(x). && (105)
\end{aligned}$$

The relations derived in the proof of Proposition 8.6 hold true for $z \le z_i$, so that the solution (U, z_i, k) satisfies (69), (70) and (72) for $z \le z_i$. Note that (65) and (66) hold true for any $(z,y) \in \{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$. We then deduce from (102) that (91) characterizes the function k on $[z_i, \infty)$. Consider (101) and take the derivative with respect to z of $U(z,\psi(z))$. One get

$$U_z(z,\psi(z)) + \psi'(z)U_y(z,\psi(z)) = 0. \tag{106}$$

Then, using (65) and (102), Equations (101) and (106) evaluated at z_i yield

$$\begin{cases} A(z_i) = \frac{p}{\psi'(z_i)}\frac{\hat{y}}{2\mu^2}h_2(\psi(z_i)), \\ B(z_i) = -\frac{p}{\psi'(z_i)}\frac{\hat{y}}{2\mu^2}h_1(\psi(z_i)). \end{cases} \tag{107}$$

We then obtain from (66) that

$$\begin{cases} A(z) = A(z_i) + \int_{z_i}^{z}\frac{\hat{y}}{2\mu^2}h_2'(k(u))\,du, \\ B(z) = B(z_i) - \int_{z_i}^{z}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du. \end{cases} \tag{108}$$

Thus, a smooth solution (U, z_i, k) to (100)-(104) satisfies $U(z,y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z,y),\ z \in \mathbb{R},\ \psi(z) < y < k(z)\}$ and the relations (70), (72), (107), (108). We prove below that such a smooth solution satisfies (105) if and only if $z_i = z_i^*$. This will imply that $k = k^*$. We have

$$U(z, \psi(x+z)) = A(z)h_1(\psi(x+z)) + B(z)h_2(\psi(x+z)) \tag{109}$$

where (67) and (70) define A and B for $z \le z_i$ and (107) and (108) define A and B for $z \ge z_i$. We deduce from (93) that

$$h_1'(k(u)) \underset{+\infty}{\sim} (\gamma-1)e^{\gamma\beta(u+x_\mu)} \quad \text{and} \quad h_2'(k(u)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}. \tag{110}$$

It follows that

$$\int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_2'(k(u))du = -\infty, \quad \int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u))du < \infty, \tag{111}$$

and also that,

$$\int_{z_i}^{z} h_2'(k(u))\, du \underset{+\infty}{\sim} \int_{z_i}^{z} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}\, du, \tag{112}$$

yielding

$$\int_{z_i}^{z} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}\, du = \frac{\gamma}{\gamma-1}\frac{1}{\beta}e^{-(\gamma-1)\beta(z+x_\mu)} - \frac{\gamma}{\gamma-1}\frac{1}{\beta}e^{-(\gamma-1)\beta(z_i+x_\mu)}.$$

A computation yields

$$\begin{aligned}
&\lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{\hat{y}}{2\mu^2} h_2'(k(u))du\; h_1(\psi(x+z)) \\
&= \frac{\gamma}{\beta(\gamma-1)}\frac{\hat{y}}{\mu^2}e^{-(\gamma-1)\beta x_\mu}e^{(\gamma-1)\beta x} - \lim_{z\longrightarrow\infty}\frac{\gamma}{\beta(\gamma-1)}\frac{\hat{y}}{\mu}e^{-(\gamma-1)\beta(z_i+x_\mu)}e^{(\gamma-1)\beta(x+z)} \\
&= \frac{\gamma}{\beta(\gamma-1)}\frac{\hat{y}}{\mu^2}e^{-(\gamma-1)\beta x_\mu}e^{(\gamma-1)\beta x},
\end{aligned} \tag{113}$$

where the last equality comes from the fact that $\gamma < 0$. We also have,

$$\int_{z_i}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du = \int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du - \int_{z}^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du$$

from which we deduce that,

$$\begin{aligned}
\lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du\; h_2(\psi(x+z)) &= \Big(\lim_{z\longrightarrow\infty} \int_{z_i}^{\infty} \frac{\hat{y}}{\mu} h_1'(k(u))\, du\; e^{-\beta\gamma(x+z)} \\
&\quad - \int_{z}^{\infty} \frac{\hat{y}}{\mu} e^{\beta\gamma(u+x_\mu)}(\gamma-1)\, du\; e^{-\beta\gamma(x+z)} \Big).
\end{aligned}$$

Thus,

$$\lim_{z\longrightarrow\infty} \int_{z_i}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du\, h_2(\psi(x+z)) = -\frac{1-\gamma}{\beta\gamma}\frac{\hat{y}}{\mu}e^{\beta\gamma x_\mu} + \lim_{z\longrightarrow\infty} e^{-\beta\gamma(x+z)} \int_{z_i}^{\infty} \frac{\hat{y}}{\mu} h_1'(k(u))\, du \tag{114}$$

Using (107), (108) and (109) together with (42), (113), (114) we obtain

$$\begin{aligned}
\lim_{z\longrightarrow\infty} U(z,\psi(x+z)) &= \lim_{z\longrightarrow\infty}\Big(\int_{z_i}^{z}\frac{\hat{y}}{2\mu^2}h_2'(k(u))\,du\; h_1(\psi(x+z))+B(z_i)h_2(\psi(x+z))\\
&\quad -\int_{z_i}^{z}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du\; h_2(\psi(x+z))\Big)\\
&= V_\mu(x)+\lim_{z\longrightarrow\infty}\left(B(z_i)h_2(\psi(x+z))-\int_{z_i}^{\infty}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du\; 2\mu e^{-\beta\gamma(x+z)}\right)\\
&= V_\mu(x)-\lim_{z\longrightarrow\infty}\left(\frac{p}{\psi'(z_i)}h_1(\psi(z_i))+\int_{z_i}^{\infty}h_1'(k(u))\,du\right)\frac{\hat{y}}{\mu}\,e^{-\beta\gamma(x+z)}.
\end{aligned}$$

Since $\gamma<0$, the function U satisfies (105) if and only if z_i satisfies (94), or equivalently if $z_i=z_i^*$. Therefore, a smooth solution (U,z_i,k) to (100)-(105) satisfies $U(z,y)=A(z)h_1(y)+B(z)h_2(y)$ on $\{(z,y),\ z\in\mathbb{R},\ \psi(z)<y<k(z)\}$ and the relations (70), (72), (107), (108) with $k=k^*$ (and thus $z_i=z_i^*$). Conversely, a computation shows that $U(z,y)=A(z)h_1(y)+B(z)h_2(y)$ on $\{(z,y),\ z\in\mathbb{R},\ \psi(z)<y<k(z)\}$ and the relations (70), (72), (107), (108) with $k=k^*$ is a solution to the system (100)-(105).

Finally, from Proposition 8.11, k_1 and k_2 in (95) are increasing so that, b^* in (96) is indeed a well defined function which is not differentiable at y^{i*} from assertion (iii) of Proposition 8.11. Then, using the change of variable $z=\phi(y)-x$, we obtain that (V,y_i^*,b^*) is a solution to the system (12)-(17). Observe that the uniqueness of the function V comes from the uniqueness of the function b^* that follows from condition (105). □

Proceeding as in the previous section, we show the following

Proposition 8.13 *The function V defined in Proposition 8.12 satisfies the assumptions of Lemma 8.9 and thus $V\geq V^*$.*

Proof of Proposition 8.13. The proof relies on arguments developed in Proposition 8.7. We first show that the function V is $\mathcal{C}^1$ on the domain $(0,\infty)\times(-\mu,\mu)$ and is $\mathcal{C}^2$ on the domain $(0,\infty)\times(-\mu,\mu)\setminus\{(b^*(y^{i*}),y^{i*})\}$. By construction, V is twice continuously differentiable on any open set in $(0,\infty)\times(-\mu,\mu)$ away from the set $\{(x,y),\ x=b^*(y)\}$. Since b^* is differentiable on $(y^*,\mu)\setminus\{y^{i*}\}$, we can proceed as in the proof of Proposition 8.7 to prove that V is of class $\mathcal{C}^2$ on $[0,\infty)\times(-\mu,\mu)\setminus\{(b^*(y^{i*}),y^{i*})\}$. Also by construction, the study of the $\mathcal{C}^1$-differentiability of V^* does not involve the derivative of b^*: to establish that V is $\mathcal{C}^1$ on $(0,\infty)\times(-\mu,\mu)$, it is sufficient to check that A and B are continuously differentiable at z_i^*. For the continuity of A (or equivalently of B) at z_i^*, observe that (70) and (91) evaluated at z_i^*, lead to (98). The differentiability of A and B at z_i^* comes from (66) and (97). Thus, V is twice differentiable almost everywhere.

Second, let us fix any $y\in(y^*,\mu)$. We deduce from the proof of Proposition 8.7 that the mapping $x\rightarrow V(x,y)$ is concave on $[0,\infty)\setminus\{\phi(y)-z_i^*\}$ if and only if

$$(k^*)'(\phi(y)-x)(h_2(k^*(\phi(y)-x)h_1(y)-h_1(k^*(\phi(y)-x)h_2(y))<0. \tag{115}$$

Note that (115) is well defined since $\phi(y)-x\neq z_i^*$ on the considered domain, such that the derivative of k^* is well defined. The reasoning developed in the proof of Proposition 8.7

shows that (115) holds true. Now, because V is linear in x outside $\{x < b^*(y)\}$ and that $V_x(b^*(y), y) = 1$ for any $y \in (-\mu, \mu)$, we deduce from the concavity of $x \to V(x, y)$ that $V_x(x, y) \geq 1$ on $[0, \infty) \setminus \{\phi(y) - z_i^*\}$. Then, because V is $\mathcal{C}^1$ on $(0, \infty) \times (-\mu, \mu)$, one obtains that $V_x(x, y) \geq 1$ on $(0, \infty) \times (-\mu, \mu)$.

Finally, the concavity and $\mathcal{C}^1$-differentiability properties together with the fact that $V_x(0, y) \leq p$ for any $y \in (-\mu, \mu)$ lead to $V_x(x, y) \leq p$ on $(0, \infty) \times (-\mu, \mu)$.

The arguments developed in the proof of Proposition 8.7 show that $\mathcal{L}V - rV \leq 0$ on $(0, \infty) \times (-\mu, \mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. Aggregating all these results, we obtain that, almost everywhere on $(0, \infty) \times (-\mu, \mu)$,

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0.$$

Observe also that V satisfies by construction $\max(-V(0, y), V_x(0, y) - p) = 0$ for all $y \in (-\mu, \mu)$.

To conclude, it remains to show that the function V has bounded first derivatives. We have $V_x(x, y) \leq p$ on $(0, \infty) \times (-\mu, \mu)$ such that $(x, y) \longrightarrow V_x(x, y)$ is bounded over $[0, \infty) \times (-\mu, \mu)$. From the expression of V in Proposition 8.12, we deduce that V_y is bounded if and only if

$$\lim_{y \longrightarrow \mu} V_y(x, y) < \infty.$$

That is, as shown in Proposition 8.7, if and only if

$$\lim_{z \longrightarrow \infty} \frac{1}{\psi'(x + z)} W_z(x, z) < \infty,$$

where $W(x, z) = V(x, \psi(x + z)) = U(z, \psi(x + z))$. Thus, we have that

$$\begin{aligned} W(x, z) &= A(z) h_1(\psi(x + z)) + B(z) h_2(\psi(x + z)) \\ &= A(z_i^*) h_1(\psi(x + z)) + B(z_i^*) h_2(\psi(x + z)) \\ &\quad + \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_2'(k(u))\, du\, h_1(\psi(x + z)) - \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du\, h_2(\psi(x + z)). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{1}{\psi'(x + z)} W_z(x, z) &= A(z_i^*) h_1'(\psi(x + z)) + B(z_i^*) h_2'(\psi(x + z)) \\ &\quad + \frac{\hat{y}}{2\mu^2} h_2'(k(z)) \frac{h_1(\psi(x + z))}{\psi'(x + z)} - \frac{\hat{y}}{2\mu^2} h_1'(k(z)) \frac{h_2(\psi(x + z))}{\psi'(x + z)} \\ &\quad + \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_2'(k(u))\, du\, h_1'(\psi(x + z)) - \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du\, h_2'(\psi(x + z)), \end{aligned}$$

or, equivalently,

$$\begin{aligned} \frac{1}{\psi'(x + z)} W_z(x, z) &= A(z_i^*) h_1'(\psi(x + z)) + B(z_i^*) h_2'(\psi(x + z)) \\ &\quad + \frac{\hat{y}\sigma^2}{2\mu^2} \frac{h_2'(k(z))}{h_2(\psi(x + z))} - \frac{\hat{y}\sigma^2}{2\mu^2} \frac{h_1'(k(z))}{h_1(\psi(x + z))} + \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_2'(k(u))\, du\, h_1'(\psi(x + z)) \\ &\quad - \int_{z_i^*}^{z} \frac{\hat{y}}{2\mu^2} h_1'(k(u))\, du\, h_2'(\psi(x + z)). \end{aligned} \tag{116}$$

Using (110), (111), (112), we have that

$$h_1'(\psi(x+z)) \underset{+\infty}{\sim} (\gamma-1)e^{\gamma\beta(x+z)}, \quad h_2'(\psi(x+z)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(x+z)},$$

$$\frac{\hat{y}\sigma^2}{2\mu^2}\frac{h_2'(k(z))}{h_2(\psi(x+z))} \underset{+\infty}{\sim} -\frac{\hat{y}}{2\mu^2}\frac{1}{\beta}\gamma e^{(1-\gamma)\beta x_\mu}e^{\gamma\beta x}e^{\beta z}, \quad \frac{\hat{y}\sigma^2}{2\mu^2}\frac{h_1'(k(z))}{h_1(\psi(x+z))} \underset{+\infty}{\sim} -\frac{\hat{y}}{2\mu^2}\frac{1}{\beta}(1-\gamma)e^{\gamma\beta x_\mu}e^{(1-\gamma)\beta x}e^{\beta z},$$

$$\int_{z_i^*}^{z} h_2'(k(u))\frac{\hat{y}}{2\mu^2}du\, h_1'(\psi(x+z)) \underset{+\infty}{\sim} \frac{\hat{y}}{2\mu^2}\gamma\frac{1}{\beta}e^{\gamma\beta x}e^{-(\gamma-1)\beta x_\mu}e^{\beta z}.$$

and

$$\begin{aligned}\lim_{z\longrightarrow\infty}\int_{z_i^*}^{z}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)) &= \lim_{z\longrightarrow\infty}\Big(\int_{z_i^*}^{\infty}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z)) \\ &- \int_{z}^{\infty}\frac{\hat{y}}{2\mu^2}h_1'(k(u))\,du\,h_2'(\psi(x+z))\Big).\end{aligned}$$

Aggregating all these relations in (116), a last computation shows that proving $\lim\limits_{y\longrightarrow\infty} V_y(x,y) < \infty$ is equivalent to prove

$$\lim_{z\longrightarrow\infty}\left(-B(z_i^*) + \frac{\hat{y}}{2\mu^2}\int_{z_i^*}^{z} h_1'(k(u))\,du\right)\gamma e^{(1-\gamma)\beta(x+z)} < \infty.$$

Noting that $\beta > 0$ and using (98), we obtain that $\lim\limits_{y\longrightarrow\infty} V_y(x,y) < \infty$ if and only if z_i^* satisfies (94), which indeed holds true by definition of z_i^* and concludes the proof of Proposition 8.13, from which it follows that $V^* \leq V$. □

The next proposition establishes the converse inequality

Proposition 8.14 *The function V can be attained by an admissible policy and thus $V \leq \overline{V}$.*

Proof of Proposition 8.14. The proof follows exactly the same arguments than those developed in Proposition 8.8. Let L^{b^*} and L^0 positive continuous increasing processes such that the process

$$\phi(Y_t) - \phi(y) + x - L_t^{b^*} + L_t^0$$

is reflected in a horizontal direction whenever $X_t = b^*(Y_t)$ and whenever $X_t = 0$. Following the results of Burdzy and Toby (1995), the processes $L_t^{b^*}$ and L_t^0 are well-defined. Then, the policies

$$D_t^* = \left((x-b^*(y))^+\mathbb{1}_{y\geq y^*} + x\mathbb{1}_{y\leq y^*}\right)\mathbb{1}_{t=0} + L_t^{b^*}\mathbb{1}_{t>0},$$

$$I_t^* = L_t^0\mathbb{1}_{Y_{\tau_0}>y_i^*}\mathbb{1}_{t>0},$$

are admissible and a computation based on Itô's formula yields

$$V(x,y) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(dD_t^* - dI_t^*)\right]$$

which concludes the proof of Proposition 8.14. □

Thus, from Propositions 8.13 and 8.14, the function V defined in Proposition 8.12 coincides with the value function V^* of problem (8). The function b^* corresponds to the dividend boundary of the shareholders' problem (8). The optimal cash reserve process is reflected along the function b^* on a horizontal direction in the (x, y)-plane. The threshold y_i^* corresponds to the level of the profitability prospects above which new shares are issued when cash reserves are depleted. To conclude the proof of our main Theorem 4.1, it remains to show the properties of the dividend boundary b^*. The next proposition yields these results.

Proposition 8.15 *The function b^* is increasing for $y \leq y^{i*}$ and decreasing for $y \geq y^{i*}$, where $y^{i*} = k^*(z_i^*)$. The function b^* attains its maximum at y^{i*}, $b^*(y^{i*}) > x_\mu$, and $\overline{y}^* > y^* > \hat{y}$.*

Proof of Proposition 8.15. Given the previous results, we only need to show that b^* is increasing over $[y^*, y^{i*}]$, decreasing over $[y^{i*}, \mu)$ and that $\overline{y}^* > y^* > \hat{y}$. First, we show that b^* is decreasing over $[y^{i*}, \mu)$. Let us introduce the notation $\tilde{b}(z) \equiv b^*(k^*(z)) = \phi(k^*(z)) - z$ with $z > z_i^*$. Because $(k^*)^{-1}$ is increasing, we deduce from the relation $b^*(y) = \tilde{b}((k^*)^{-1}(y))$ that, for any $z > z_i^*$, $\tilde{b}'(z) = (k^*)'(z)\phi'(k^*(z)) - 1$ has the same sign than $b^{*\prime}(y)$ for any $y > y^{i*}$. In addition, the relation $k^*(z) = \psi(\tilde{b}(z) + z)$ leads to $(k^*)'(z) = (1 + \tilde{b}'(z))\psi'(z + \tilde{b}(z))$, that is $\tilde{b}'(z) = \frac{k^{*\prime}(z)}{\psi'(z+\tilde{b}(z))} - 1$. Therefore, to show that b^* is decreasing over $[y^{i*}, \mu)$, we prove that $\frac{k^{*\prime}(z)}{\psi'(z+\tilde{b}(z))} - 1 > 0$ for any $z > z_i^*$. This latter inequality follows from a computation developed in Lemma 8.17 in the additional appendix. Then, since b^* is decreasing on $[y^{i*}, \mu)$ and $b^*(\mu) = x_\mu$, we have that $b^*(y^{i*}) > x_\mu$. It remains to show that b^* is increasing over $[y^*, y^{i*}]$ and that $\overline{y}^* > y^* > \hat{y}$. We obtain from our previous results that

$$V(x, y; b^*) = V^*(x, y) \geq \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} d\overline{D}_t\right] = V(x, y; \overline{b}),$$

where $V(x, y; b^*)$ is defined in Proposition 8.12, and $V(x, y; \overline{b})$ is defined in Proposition 8.6. Equation (89) defines the process $\overline{D}$. In particular we have that

$$V(b^*(\overline{y}^*), \overline{y}^*; b^*) \geq V(\overline{b}(\overline{y}^*), \overline{y}^*; \overline{b}) = V(0, \overline{y}^*; \overline{b}),$$

which implies $b^*(\overline{y}^*) \geq \overline{b}(\overline{y}^*)$. Let us recall that, for $y \leq y^{i*}$ the functions b^* and $\overline{b}$ satisfy (52) and (54). Then, the non crossing property of ordinary differential equations implies $b^*(y) \geq \overline{b}(y)$, for $y \leq y^{i*}$. It then follows from the proof of Lemma 8.5 that the function b^* is increasing for $y \leq y^{i*}$ and that $\overline{y}^* \geq y^*$. The same reasoning than in the proof of Lemma 8.5 shows that $y^* > \hat{y}$. Finally, a computation shows that y^{i*} is increasing in the proportional issuance cost p. Recalling that $V(x, y; b^*) = V(x, y; \overline{b})$ for $p = \overline{p}$, we then deduce that $\overline{y}^* > y^*$ for $1 < p < \overline{p}$. This concludes the proof of Proposition 8.15. The proof of Theorem 4.1 is complete. □

Proofs of Lemma 8.9 and Lemma 8.4. The proof of Lemma 8.4 follows from a straightforward adaptation of the proof of Lemma 8.9 that we show below.

Let us consider a pair of admissible policies D and I and let us write $D_t = D_t^c + D_t^d$, $I_t = I_t^c + I_t^d$, where D_t^c (resp. I_t^c) is the continuous part of D_t (resp. I_t) and D_t^d (resp. I_t^d) is the pure discontinuous part of D_t (resp. I_t). We recall the dynamics of the cash reserve process and of the profitability prospects process:

$$dX_t = Y_t\,dt + \sigma dB_t - dD_t + \frac{dI_t}{p} \text{ and, } dY_t = \frac{\mu^2 - Y_t^2}{\sigma}\,dB_t.$$

Applying the generalized Itô's formula to a function V that satisfies the assumptions of Lemma 8.9, we can write for $\tau_0 = \inf\{t \geq 0,\, X_t = 0\}$,

$$\begin{aligned} e^{-r(t\wedge\tau_0)}V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0}) &= V(x,y) + \int_0^{t\wedge\tau_0} e^{-rs}\left(\mathcal{L}V(X_{s-},Y_s) - rV(X_{s-},Y_s)\right)\,ds \\ &+ \int_0^{t\wedge\tau_0} e^{-rs}V_x(X_{s-},Y_s)\sigma dB_s + \int_0^{t\wedge\tau_0} e^{-rs}V_y(X_{s-},Y_s)\frac{\mu^2 - Y_s^2}{\sigma}dW_s \\ &- \int_0^{(t\wedge\tau_0)} e^{-rs}V_x(X_s,Y_s)\,dD_s^c - \int_0^{(t\wedge\tau_0)} e^{-rs}V_x(X_s,Y_s)\,\frac{dI_s^c}{p} \\ &+ \sum_{s\leq t\wedge\tau_0} e^{-rs}(V(X_s,Y_s) - V(X_{s-},Y_s))(\mathbb{1}_{(\Delta X)_s>0} + \mathbb{1}_{(\Delta X)_s<0}), \end{aligned}$$

where $(\Delta X)_s = X_s - X_{s^-}$. By assumption, the second term of the right-hand side is negative and, because V has bounded first derivatives, the two stochastic integrals are centered square integrable martingales. Taking expectations and using $1 \leq V_x \leq p$, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{t\wedge\tau_0}, Y_{t\wedge\tau_0})\right] &\leq V(x,y) - \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,dD_s^c\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,\frac{dI_s^c}{p}\right] \\ &+ \mathbb{E}\left[\sum_{s\leq t\wedge\tau_0} e^{-rs}(V(X_s,Y_s) - V(X_{s-},Y_s)(\mathbb{1}_{(\Delta X)_s>0} + \mathbb{1}_{(\Delta X)_s<0}))\right]. \end{aligned}$$

Observe that there are two types of jumps for the cash reserve process $(X_t)_{t\geq 0}$. When there is a dividend distribution, $(\Delta X)_s < 0$ so that $X_{s-} - X_s = D_s - D_{s-} > 0$. When there is an issue of shares, $(\Delta X)_s > 0$ and $X_s - X_{s-} = \frac{I_s - I_{s-}}{p} > 0$. Therefore, the Mean-Value theorem gives the existence of a random variable $\theta \in [X_s, X_{s-}]$ when $X_{s-} - X_s = D_s - D_{s-} > 0$ and a random variable $\eta \in [X_{s-}, X_s]$ when $X_s - X_{s-} = \frac{I_s - I_{s-}}{p} > 0$ such that:

- on the set $\{X_{s-} - X_s = D_s - D_{s-}\}$, $V(X_s,Y_s) - V(X_{s-},Y_s) = -V_x(\theta,Y_s)(X_{s-} - X_s)$,
- on the set $\{X_s - X_{s-} = \frac{I_s - I_{s-}}{p}\}$, $V(X_s,Y_s) - V(X_{s-},Y_s) = V_x(\eta,Y_s)(X_s - X_{s-})$.

We deduce that,

$$\begin{aligned} V(X_s,Y_s) - V(X_{s-},Y_s) &= V_x(\eta,Y_s)(\frac{I_s - I_{s-}}{p})\mathbb{1}_{(\Delta X)_s>0} - V_x(\theta,Y_s)(D_s - D_{s-})\mathbb{1}_{(\Delta X)_s<0} \\ &\leq \frac{I_s - I_{s-}}{p}\mathbb{1}_{(\Delta X)_s>0} - (D_s - D_{s-})\mathbb{1}_{(\Delta X)_s<0}. \end{aligned}$$

Finally, we obtain

$$V(x,y) \geq \mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{(t\wedge\tau_0)},Y_{(t\wedge\tau_0)})\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_0} e^{-rs}\,(dD_s - \frac{dI_s}{p})\right].$$

In order to get rid of the first term of the right-hand side, we observe that under the assumptions of Lemma 8.9, we have $V(x,y) \leq V(0,y) + px$ that implies

$$\mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{(t\wedge\tau_0)},Y_{(t\wedge\tau_0)})\right] \leq e^{-rt}V_\mu(0) + p\mathbb{E}[e^{-rt}X_t].$$

Letting t go to ∞ yields

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-r(t\wedge\tau_0)}V(X_{(t\wedge\tau_0)},Y_{(t\wedge\tau_0)})\right] = 0,$$

and thus

$$V(x,y) \geq \mathbb{E}\left[\int_0^{\tau_0} e^{-rs}\,(dD_s - \frac{dI_s}{p})\right],$$

which ends the proof of Lemma 8.9. □

Proof of Lemma 8.5. Observe that the function f defined in (54) does not satisfy the Lipschitz condition on an open domain containing (x,μ) with $x \geq 0$, so that the existence and uniqueness of a solution to (52), (53) require a specific analysis.

We remark that, the denominator of (54) is strictly positive if and only if $x > l_1(y)$ where

$$l_1(y) = \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}).$$

Thus, f satisfies a local Lipschitz condition with respect to x in $\mathcal{D}$, where $\mathcal{D} = \{(x,y) \in \mathbb{R}\times(-\mu,\mu)\,|\,x > l_1(y)\}$. Thus, for any (x,y), there exists a unique solution $g_{x,y}$ to (52) defined on a maximal interval $I \subset (-\mu,\mu)$ passing through (x,y).

Second, the numerator of (54) is strictly positive if and only if[32] $x > l_2(y)$ where,

$$l_2(y) = \frac{\hat{y}}{\mu}\phi\left(\frac{y\hat{y}}{\mu - r\sigma^2(\frac{\hat{y}}{\mu})^2\frac{1}{\mu}}\right).$$

The function l_2 is continuously increasing on $[-\mu,\mu]$ and satisfies the inequality $l_2(y) > l_1(y)$ for any $y \in (0,\mu]$. Furthermore, $l_1(0) = l_2(0) = 0$ and $l_1(\mu) < l_2(\mu) = \overline{x}_\mu$. To see the last equality, use (39) and remark that $2\phi(\hat{y}) = \phi(\frac{\hat{y}}{1-r\sigma^2(\frac{\hat{y}}{\mu})^2\frac{1}{\mu^2}})$. This leads to $l_2(\mu) = \overline{x}_\mu$.

We deduce from the above observations, that any solution g to (52) entering in the domain $\{(x,y) \in \mathcal{D}\,|\,l_1(y) < x < l_2(y)\}$ remains in this domain. Since l_2 is bounded above by $\overline{x}_\mu$ on $[-\mu,\mu]$, it follows also that any solution g to (52) defined on a maximal interval I and passing through $(x_0,y_0) \in \{(x,y) \in \mathcal{D}\,|\,x \geq \overline{x}_\mu\}$ is strictly increasing and satisfies $g(y) > l_2(y)$ for all $y \in I$.

[32] Note that, the definition of $\hat{y}$ in (35) implies that $\mu - r\sigma^2(\frac{\hat{y}}{\mu})^2\frac{1}{\mu} = \mu(1 - r\sigma^2\frac{1}{\mu^2+2r\sigma^2}) > 0$.

Now, let $(y_n)_{n\geq 0}$ an increasing sequence converging to μ. For each $n \in \mathbb{N}$ there exists a unique solution $g_{\overline{x}_\mu, y_n}$ to (52) satisfying $g_n(\overline{x}_\mu) = y_n$. Let us consider the sequence of functions $(g_n)_{n\geq 0}$ defined by the relations

$$\begin{cases} g_n(y) = g_{\overline{x}_\mu, y_n}(y) & \forall y \in (0, y_n], \\ g_n(y) = \overline{x}_\mu & \forall y \in [y_n, \mu]. \end{cases}$$

Our previous remarks on the solutions to (52) together with a standard non crossing property yield that, $(g_n)_{n\geq 0}$ is a decreasing sequence of increasing functions defined on $(0, \mu]$ and bounded above by $\overline{x}_\mu$. Thus, it admits a pointwise limit g defined on $(0, \mu]$. The function g is bounded above by $\overline{x}_\mu$ and satisfies $g(\mu) = \overline{x}_\mu$. We show below that g satisfies (52).

By construction, for each $n \in \mathbb{N}$, for any $y \in (0, \mu)$ one has

$$g_n(y) = \overline{x}_\mu - \int_y^\mu f(g_n(s), s)\, \mathbb{1}_{s\leq y_n}\, ds.$$

A direct computation shows that, for any fixed $y > 0$, the mapping $x \longrightarrow f(x, y)$ is continuously increasing over $\{x \,|\, x \geq l_2(y)\}$. We deduce that, for any $y \in (0, \mu)$,

$$\int_y^\mu \lim_{n\longrightarrow\infty} f(g_n(s), s)\, \mathbb{1}_{s\leq y_n}\, ds = \int_y^\mu f(g(s), s)\, ds \leq \int_y^\mu f(\overline{x}_\mu, s)\, ds < \infty,$$

where the last inequality comes from the fact that the mapping $s \longrightarrow f(\overline{x}_\mu, s)$ is continuous over $(0, \mu)$ with $\lim_{s\longrightarrow\mu} f(\overline{x}_\mu, s) = \frac{1}{2}\frac{1 - r\sigma^2(\frac{\hat{y}}{\mu})^2\frac{1}{\mu^2}}{r(\frac{\hat{y}}{\mu})^2} < \infty$. It results from the dominated convergence Theorem that,

$$g(y) = \overline{x}_\mu - \int_y^\mu f(g(s), s)\, ds. \tag{117}$$

Thus, g is defined and increasing on $(0, \mu]$, satisfies the ode (52)-(53). A standard extension argument ensures that g is defined on a maximal interval $I \subset (-\mu, \mu)$ as well.

We show that $\overline{y}^* \equiv g^{-1}(0)$ is well defined and satisfies $\overline{y}^* > \hat{y}$. Take the solution $g_{0,\hat{y}}$ to (52) defined on a maximal interval $I \subset (-\mu, \mu)$ passing through $(0, \hat{y})$. A computation shows that the function $v_1(y) = \frac{\hat{y}}{\mu}(\phi(\hat{y}) - \phi(y))$ defined on $(-\mu, \mu)$ satisfies $v_1(\hat{y}) = g_{0,\hat{y}}(\hat{y}) = 0$ together with the inequality $v_1'(y) < f(v_1(y), y)$ for any $y \in (-\mu, 0]$. We deduce that $g_{0,\hat{y}}(y) > v_1(y)$ for all $y \in (\hat{y}, 0]$. From the Cauchy-Lipschitz Theorem it follows that $g_{0,\hat{y}} > g_{v_1(0),0}$ on a maximal interval, where $g_{v_1(0),0}$ is the solution to (52) passing through $(v_1(0), 0)$.

Now, let us consider the function $v_2(y) = \frac{\hat{y}}{\mu}(\phi(y\frac{\hat{y}}{\mu}) + \phi(\hat{y})) \geq l_2(y)$ on $[0, \mu]$. Computations shows that $v_2(0) = v_1(0)$, $v_2(\mu) = \overline{x}_\mu$ and $v_2'(y) \leq f(v_2(y), y)$ for any $y \in [0, \mu]$. We deduce that $g_{0,\hat{y}}(y) \geq v_2(y)$ for any $y \in [0, \mu]$. It follows that $g_{0,\hat{y}} > g_{v_1(0),0} \geq g$ which implies that $\overline{y}^* \equiv g^{-1}(0) > \hat{y}$.

Finally, we show that g is the unique solution to $g'(y) = f(g(y), y)$ satisfying the boundary condition $g(\mu) = \overline{x}_\mu$. Suppose the contrary, let g and $\tilde{g}$ be two solutions to (52) with

$g(\mu) = \tilde{g}(\mu) = \overline{x}_\mu$ and $\tilde{g}(y) > \tilde{g}(y)$ over $(0, \mu)$. The functions g and $\tilde{g}$ satisfy (117). It follows that,

$$\tilde{g}(y) - g(y) \;=\; \int_y^\mu f(g(s), s) - f(\tilde{g}(s), s)\, ds. \tag{118}$$

The right hand side of (118) is strictly positive whereas its left hand side is negative because the mapping $x \longrightarrow f(x, y)$ is increasing for any fixed $y > 0$, thus, a contradiction.

We now turn to the study of the function $\overline{k} = (\phi - \overline{b})^{-1} : [\phi(\overline{y}^*), \infty) \longrightarrow [\overline{y}^*, \mu)$. We observe that

$$\begin{aligned}
\phi'(y) - g'(y) \;&=\; \frac{\sigma^2}{\mu^2 - y^2}\left(1 - \frac{y\hat{y} - \left(\mu - r\sigma^2\left(\frac{\hat{y}}{\mu}\right)^2 \frac{1}{\mu}\right)\psi\left(\frac{\mu}{\hat{y}}\, g(y)\right)}{y\hat{y} \;-\; \mu\psi\left(\frac{\mu}{\hat{y}}\, g(y)\right)}\right) \\
&=\; \frac{\sigma^2}{\mu^2 - y^2}\, \frac{\frac{1}{\mu} r\sigma^2 (\frac{\hat{y}}{\mu})^2 \psi\left(\frac{\mu}{\hat{y}}\, g(y)\right)}{-y\hat{y} + \mu\psi\left(\frac{\mu}{\hat{y}}\, x\right)}
\end{aligned}$$

is positive for $g(y) > \max(0, \frac{\hat{y}}{\mu}\phi(g(y)\frac{\hat{y}}{\mu})$, so that $\phi - \overline{b}$ is strictly increasing over $[\overline{y}^*, \mu)$ where $\overline{b} = \max(0, g)$ with g satisfying (52) and (54). Thus, $\overline{k} = (\phi - \overline{b})^{-1} : [\phi(\overline{y}^*), \infty) \longrightarrow [\overline{y}^*, \mu)$ is well defined, increasing and satisfies

$$\overline{k}'(z) = \frac{1}{(\phi - \overline{b})'(\overline{k}(z))} = \frac{\mu^3}{\hat{y} r\sigma^2}\frac{\mu^2 - k(z)^2}{\sigma^2}\frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(\overline{k}(z)) - z)\right)\mu - y\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(\overline{k}(z)) - z)\right)}.$$

Because $\overline{b} > 0$ and $\overline{k} = (\phi - \overline{b})^{-1}$ we have that $\phi(\overline{k}(z)) - z > 0$. Because $\overline{b}(y) > \frac{\hat{y}}{\mu}\phi(\overline{b}(y)\frac{\hat{y}}{\mu}$ we have that $\psi\left(\frac{\mu}{\hat{y}}(\phi(\overline{k}(z)) - z)\right)\mu - y\hat{y} > 0$, so that $(z, \overline{k}(z)) \in \widetilde{\mathcal{D}} = \{(z, y) \in \mathbb{R} \times (-\mu, \mu) \,|\, \phi(y) - z > \max(0, \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}))\}$. Finally, the relation $\overline{k} = (\phi - \overline{b})^{-1}$ together with $\lim\limits_{y \longrightarrow \mu} \overline{b}(y) = \overline{b}(\mu) = \overline{x}_\mu < \infty$ implies that $\lim\limits_{y \longrightarrow \mu} \overline{k}^{-1}(y) = \infty$ and in turn that $\lim\limits_{z \longrightarrow \infty} \overline{k}(z) = \mu$ since $\overline{k}$ is increasing. Then, posing $y = \overline{k}(z)$, the relation $\overline{b}(y) = \phi(y) - \overline{k}^{-1}(y)$ leads to $\lim\limits_{z \longrightarrow \infty} \overline{b}(k(z)) = \phi(\overline{k}(z)) - z = \overline{x}_\mu$.

Conversely, let us consider a C^1-function k solution to the ordinary differential equation (55), (56) on the domain $\widetilde{\mathcal{D}}$. The function k is increasing so that, $b = \phi - k^{-1}$ is well defined over $(\overline{y}^*, \mu)$. It is positive since by assumption $(z, k(z)) \in \widetilde{\mathcal{D}}$. A direct computation shows that b satisfies the ordinary differential equation (55), (56) on the domain $\mathcal{D}$. Thus, we obtain that $b = \overline{b}$ and in turn $k = \overline{k}$. The proof of Lemma 8.5 is complete. $\square$

Proof of Lemma 8.10. Since for any y fixed in $(-\mu, \mu)$, the mapping $x \longrightarrow \overline{V}_x(x, y)$ is concave on $[0, \infty)$, we only have to check that $\overline{V}_x(0, y) < \overline{p}$ for any $y \in (-\mu, \mu)$. Noting that $\overline{V}'_\mu(0) = \overline{p}$ and that $\lim\limits_{y \longrightarrow \mu} \overline{V}_x(0, y) = \overline{V}'_\mu(0)$, the result follows from the fact that the mapping $y \longrightarrow \overline{V}_x(0,y)$ is increasing.[33] To see that latter point, note that $\overline{V}(x,y) = U(\phi(y) - x, y)$ where the function U is defined in the proof of Proposition 8.6. We obtain that $\overline{V}_x(0,y) = -U_z(0,\phi(y))$ and thus that $\frac{d}{dy}\overline{V}_x(0,y) = -\phi'(y)U_{zz}(0,\phi(y))$. The result follows since ϕ is increasing and $U_{zz}(z,y) = \frac{\hat{y}}{2\mu^2}\overline{k}'(z)(h_2''(\overline{k}(z))h_1(y) - h_1''(\overline{k}(z))h_2(y))$ which we know to be negative from the proof of the concavity of the mapping $x \longrightarrow \overline{V}(x,y)$ in Proposition 8.7. □

Proof of Proposition 8.11.

Proof of Assertion (i). A computation shows that the mapping

$$y \longrightarrow G(z,y) = -h_1'(y)h_2(\psi(z)) + h_2'(y)h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p$$

is well defined over $[\psi(z),\infty)$, increasing and satisfies the equality $G(\psi(z),z) = \frac{2\mu^2}{\hat{y}}(p-1) < 0$ and $\lim_{y\longrightarrow\mu} G(z,y) = +\infty$. Thus, the function k is well defined over $[z_i,\infty)$ and satisfies the inequality $k > \psi$. Applying the implicit function Theorem, we deduce from the relation $G(z,k(z)) = 0$ that, the function k is differentiable and satisfies for any $z > z_i$

$$k'(z) = \psi'(z)\frac{h_2'(k(z))h_1'(\psi(z)) - h_1'(k(z))h_2'(\psi(z))}{h_1''(k(z))h_2(\psi(z)) - h_2''(k(z)h_1(\psi(z))}. \tag{119}$$

We saw in the proof of Proposition 8.7 that the denominator of the right hand side of (119) is negative. The numerator of k' in (119) is also negative. To see this point, remark that $x \longrightarrow h_2'(x)h_1'(y) - h_1'(x)h_2'(y)$ is decreasing over $[y,\mu)$ since it takes the value 0 at $x = y$, and its derivative has the same sign as $h_2(x)h_1'(y) - h_1(x)h_2'(y) < 0$. Thus, k is increasing over $[z_i,\infty)$. Then, assertion (i) of Propostion 8.11 follows from the next lemma.

Lemma 8.16 *the following holds*

(i) $\psi(z + x_\mu) < k(z)\ \ \forall z$,

(ii) $\psi(z + x_\mu + \epsilon) > k(z)\ \ \forall\epsilon > 0$ *for z sufficiently large.*

Proof of Lemma 8.16. We show that, for any z, $G(z,\psi(x_\mu + z)) < 0$ and that, for any $\epsilon > 0$ and any z sufficiently large, $G(z,\psi(x_\mu + z + \epsilon)) > 0$. Computations yield, for $x \geq 0$

$$G(z,\psi(x+z)) = g_1(x) + e^{-\beta z}g_2(x),$$

with

$$\begin{aligned} g_1(x) &= (-\frac{\mu^2}{\hat{y}} + \mu)e^{\beta\gamma x} - (\frac{\mu^2}{\hat{y}} + \mu)e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}, = 2(1-\gamma)\mu e^{\beta\gamma x} - 2\gamma\mu e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}. \\ g_2(x) &= (-\frac{\mu^2}{\hat{y}} + \mu)e^{-\beta\gamma x} - (\frac{\mu^2}{\hat{y}} + \mu)e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}} \\ &= 2(1-\gamma)\mu e^{-\beta\gamma x} - 2\gamma\mu e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}. \end{aligned} \tag{120}$$

[33] A computation based on (73) and (75) yields $\lim_{y\longrightarrow\mu}\overline{V}_x(0,y) = \overline{V}_\mu'(0)$.

To prove (i) and (ii), we show that the functions g_1 and g_2 are increasing and satisfiy $g_2(x_\mu) < g_1(x_\mu) = 0$. A computation leads to

$$g_1'(x) > 0 \Leftrightarrow e^{(1-2\gamma)\beta x} > 1, \text{ and } g_2'(x) > 0 \Leftrightarrow e^{-(1-2\gamma)\beta x} < 1.$$

Both inequalities hold true since $(1-2\gamma)\beta > 0$. The relation $g_1(x_\mu) = 0$ follows from the definition of x_μ. Finally, using (120) and (43) and rearrangging terms yield that

$$g_2(x_\mu) < 0 \Leftrightarrow \overline{g}(x_\mu) > \underline{g}(x_\mu) \tag{121}$$

with $\overline{g}(x) = (1-\gamma)(e^{\gamma\beta x} - e^{-\beta\gamma x})$ and $\underline{g}(x) = \gamma(e^{(1-\gamma)\beta x} - e^{(\gamma-1)\beta x})$. It is easy to see that $\overline{g}(0) = \underline{g}(0) = 0$ and that $\overline{g}$ and $\underline{g}$ are decreasing. To prove (121), we remark that, for any $x > 0$, $\overline{g}'(x) > \underline{g}'(x)$. Indeed, this latter inequality is equivalent to

$$e^{\gamma\beta x} + e^{-\beta\gamma x} < e^{\beta(\gamma-1)x} + e^{\beta(1-\gamma)x},$$

which, given that $-\beta\gamma < \beta(1-\gamma)$, follows the properties of the function cosh. □

Thus, from (i) and (ii) we have that,

$$\forall \epsilon > 0, \ \exists \overline{z}, \ \forall z \geq \overline{z}, \ \ \psi(z + x_\mu) < k(z) < \psi(z + x_\mu + \epsilon).$$

Assertions (92) and (93) follow by noting that, $k(z) = \psi((\phi(k(z)) - z) + z)$ and that ψ is a bounded continuous and increasing function. The proof of assertion (i) is complete.

Proof of Assertion (ii). We start with the existence of a solution z_i^* to equation (94). Let us consider

$$f(z) = \frac{p}{\psi'(z)} h_1(\psi(z)) + \int_z^\infty h_1'(k(u))\, du. \tag{122}$$

To begin, we show that $f(z_i^*) > 0$ and that $\lim_{z\longrightarrow\infty} f(z) < 0$. Since function h_1' is negative and increasing, we deduce from the inequality $\psi < k$ that

$$\int_{z_i^*}^\infty h_1'(\psi(u))\, du < \int_{z_i^*}^\infty h_1'(k(u))\, du. \tag{123}$$

Thus, to show that $f(z_i^*) > 0$, we show that

$$\frac{p}{\psi'(z_i^*)} h_1(\psi(\hat{z})) + \int_{z_i^*}^\infty h_1'(\psi(u))\, du > 0.$$

Computations are explicit and yield that (123) is equivalent to

$$p(1 + e^{-\beta z_i^*}) - \frac{\gamma}{\gamma - 1} e^{-\beta z_i^*} + \frac{1-\gamma}{\gamma} > 0. \tag{124}$$

An easy computation shows that the left hand side of (124) is equal to zero when $p = 1$. This implies that $f(z_i^*) > 0$ for $p > 1$. We already know that $k = \psi$ when $p = 1$. We thus obtained as a by product result that $z_i^* = \hat{z}$ when $p = 1$.

We show that $\lim_{z \longrightarrow \infty} f(z) = 0^-$. From (93), it is sufficient to show that,

$$\frac{p}{\beta} e^{\beta\gamma z}(1 + e^{-\beta z}) + \int_z^\infty h_1'(\psi(u + x_\mu))\, du < 0 \Leftrightarrow p(1 + e^{-\beta z}) < \frac{1-\gamma}{-\gamma} e^{\beta\gamma x_\mu}.$$

This latter inequality follows from (45), thus the result. Therefore, there exists z_i^* such that $f(z_i^*) = 0$.

Uniqueness of z_i^*. A direct computation shows that $f'(z) < 0$ for $z < 0$. Therefore, given that $\lim_{z \longrightarrow \infty} f(z) < 0$, if f has more than one zero, there exists z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ and $f'(z_2) < 0$. We reason by way of contradiction and prove that if there are z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ then $f'(z_2) > 0$ which contradicts $\lim_{z \longrightarrow \infty} f(z) < 0$.

We consider below $g(z) = f(z)h_2(\psi(z))$ that as the same zeros and the same sign than f. We have that

$$g(z) = p\sigma^2 + \int_z^\infty h_1'(k(u))\, du\; h_2(\psi(u)).$$

Thus,

$$\begin{aligned} g'(z) &= \psi'(z)h_2'(\psi(z)) \int_z^\infty h_1'(k(u))\, du - h_2(\psi(z))h_1'(k(z)) \\ &= \psi'(z)h_2'(\psi(z)) \int_z^\infty h_1'(k(u))\, du - p\frac{2\mu^2}{\hat{y}} - h_1(\psi(z))h_2'(k(z)), \end{aligned}$$

where the last equality follows from (91). By assumption $g(z_1) = 0$ and $g'(z_1) > 0$, using (122) we get that

$$g'(z_1) - ph_2'(\psi(z_1))h_1(\psi(z_1)) - p\frac{2\mu^2}{\hat{y}} - h_1(\psi(z_1))h_2'(k(z_1)) > 0.$$

Any zero z of g satisfies

$$g'(z) = -ph_2'(\psi(z))h_1(\psi(z)) - p\frac{2\mu^2}{\hat{y}} - h_1(\psi(z))h_2'(k(z)).$$

We show that

$$q_1 :\ z \longrightarrow -h_2'(\psi(z))h_1(\psi(z)) \ \text{ and, } \ q_2 :\ z \longrightarrow -h_1(\psi(z))h_2'(k(z))$$

are increasing functions which will imply that $g'(z_2) > 0$. We have

$$sign\{q_1'(z))\} = sign\{-2r\sigma^2 - (\frac{\mu^2}{\hat{y}} - \psi(z))(-\frac{\mu^2}{\hat{y}} - \psi(z))\} = sign\{\mu^2 - \psi^2(z)\} > 0$$

Also, we have

$$q_2'(z) = -k'(z)h_2''(k(z))h_1(\psi(z)) - h_1'(\psi(z))\psi'(z)h_2'(k(z)).$$

It follows that

$$q_2'(z) \geq 0 \Leftrightarrow k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} < \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{\hat{y}} - k(z))}{-h_2(k(z))h_1(\psi(z))}. \tag{125}$$

Using (119), a computation yields that

$$k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} = \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{\hat{y}} - k(z)) - h_1(k(z))h_2'(\psi(z))(\frac{\mu^2}{\hat{y}} - k(z))}{h_1(k(z))h_2(\psi(z)) - h_2(k(z))h_1(\psi(z)}. \tag{126}$$

Using (125) and (126), a computation shows that $q_2'(z) \geq 0$ is equivalent to $\psi(z) < k(z)$, which we know to be true, thus the result.

Proof of Assertion (iii). Given assertion (i), we only need to prove that the solution k_1 to the ordinary differential equation

$$\begin{aligned} k_1'(z) &= \Theta(z, k_1(z)), \\ k_1(z_i^*) &= k_2(z_i^*), \end{aligned}$$

with

$$\Theta(z, y) = \frac{\mu^3}{\hat{y}r\sigma^2}\frac{\mu^2 - y^2}{\sigma^2}\frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right)\mu - y\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right)}$$

defined on the domain $\widetilde{\mathcal{D}} = \{(z, y) \in \mathbb{R} \times (-\mu, \mu) \,|\, \phi(y) - z > \max(0, \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}))\}$, is a well defined continuously differentiable and increasing function over $(-\infty, z_i^*)$. We deduce from the proof of Lemma 8.5 that the ode

$$\begin{aligned} k'(z) &= \Theta(z, k(z)), \\ k(z_i) &= y_i, \end{aligned}$$

where the couple (z_i, y_i) satisfies

$$\phi(y_i) - z_i > \max(0, \frac{\hat{y}}{\mu}\phi(y_i\frac{\hat{y}}{\mu})), \tag{127}$$

has a unique solution that is continuously differentiable and increasing over $(-\infty, z_i)$. To establish Assertion (iii), it thus remain to show that the couple $(z_i^*, k_2(z_i^*))$ satisfies (127). We deduce from (71) and (72) that this requirement is equivalent to

$$h_1'(k_2(z_i^*))h_2(\psi(z_i^*)) + h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) > 0. \tag{128}$$

Consider first the case $z_i^* \leq 0$. Let us recall that the mapping $y \longrightarrow h_1'(y)h_2(\psi(z)) + h_2'(y)h_1(\psi(z))$ is increasing and that $k_2 > \psi$. It follows that

$$h_1'(k_2(z_i^*))h_2(\psi(z_i^*)) + h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) > h_1'(\psi(z_i^*))h_2(\psi(z_i^*)) + h_2'(\psi(z_i^*))h_1(\psi(z_i^*)).$$

The latter term is equal to $-2\psi(z_i^*)$ which is strictly positive, thus, (128) is satisfied. Next, consider that $z_i^* > 0$. The function k_2 satisfies (91). It follows that (128) is equivalent to $h_2'(k_2(z_i^*))h_1(\psi(z_i^*)) + \frac{\mu^2}{\hat{y}}p > 0$. From the proof of assertion (ii) we know that z_i^* satisfies

$$p\frac{\mu^2}{\hat{y}} + h_1(\psi(z_i^*))h_2'(k(z_i^*)) > -ph_2'(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{\hat{y}}.$$

Then, a direct computation shows that $-ph_2'(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{\hat{y}} > 0$ if and only if $\psi(z_i^*) > 0$ which is true since we consider $z_i^* > 0$. As a final remark, an easy computation based on (71) and (119) yields that $k_1'(z) \neq k_2'(z)$ so that k^* is not differentiable at z_i^*. The proof of assertion (iii) is complete. □

The next Lemma completes the proof of Proposition 8.15.

Lemma 8.17 *The following holds*

$$\frac{{k^*}'(z)}{\psi'(z+\tilde{b}(z))} - 1 > 0 \quad \text{for any } z > z_i^*.$$

Proof of Lemma 8.17. From Proposition 8.11 and Proposition 8.14, we deduce that k^* satisfies (119). Therefore, to prove Lemma 8.17, we show that

$$\frac{\psi'(z)}{\psi'(z+x)}\frac{h_2'(\psi(z+x))h_1'(\psi(z)) - h_1'(\psi(z+x))h_2'(\psi(z))}{h_1''(\psi(z+x))h_2(\psi(z)) - h_2''(\psi(z+x))h_1(\psi(z))} - 1 < 0.$$

Computations yield that the latter inequality is equivalent to

$$h_1(\psi(z+x))h_2(\psi(z))\left(\psi'(z)(-\frac{\mu^2}{\hat{y}} - \psi(x+z))(\frac{\mu^2}{\hat{y}} - \psi(z)) - \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\right) +$$
$$h_2(\psi(z+x))h_1(\psi(z))\left(\psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}\right) > 0. \tag{129}$$

We remark that

$$\psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} > 0, \tag{130}$$

and

$$h_2(\psi(z+x))h_1(\psi(z)) > h_1(\psi(z+x))h_2(\psi(z)). \tag{131}$$

A computation shows that the sign of

$$\psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}$$
$$+\psi'(z)(-\frac{\mu^2}{\hat{y}} - \psi(x+z))(\frac{\mu^2}{\hat{y}} - \psi(z)) - \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))}$$

is the same as the sign of $-\psi(z) + \psi(x+z)$ which is positive. It then follows from (129), (130) and (131) that the mapping $y \longrightarrow b^*(y)$ is decreasing over $[y^{i*}, \mu)$.