# Dynamic Contracting with Many Agents

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## Abstract

We analyze dynamic capital allocation and risk sharing between a principal and many agents, who privately observe their output. The state variables of the mechanism design problem are aggregate capital and the distribution of continuation utilities across agents. This gives rise to a Bellman equation in an infinite dimensional space, which we solve with mean-field techniques. We fully characterize the optimal mechanism and show that the level of risk agents must be exposed to for incentive reasons is decreasing in their initial outside utility. We extend classical welfare theorems by showing that any incentive-constrained optimal allocation can be implemented as an equilibrium allocation, with appropriate transfers and wealth taxation by the principal.

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# 1 Introduction

How should capital be allocated and risks shared in a dynamic production economy without aggregate risk? In the absence of informational frictions, the answer is clear: capital should be allocated according to individual productivities and risks should be eliminated by diversification. However, when information about individual outputs is private, incentive compatibility constraints must be taken into account. This paper studies how these constraints affect capital accumulation and risk sharing.

To address these issues, we consider an infinite horizon economy with a large number of risk averse agents and a single good that can be consumed or invested as capital. Each agent operates a project whose output is proportional to the amount of capital under his/her management and subject to idiosyncratic shocks. Individual unit outputs are i.i.d. so that a version of the law of large numbers applies, implying that aggregate output is deterministic.

We assume agents privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [10].[1] In contrast to output and consumption, capital is observable. Applying the revelation principle, we study the revelation mechanism, in which agents truthfully report their output to the principal, who then allocates consumption and capital according to the reports. The dynamic optimal mechanism allocates capital and consumption to maximize the principal's utility, subject to the participation and incentive constraints of the agents and the aggregate resource constraint.[2]

To provide agents with incentives not to divert output, the optimal contract specifies an increase (resp. decrease) of consumption **and** capital for agents whose output is larger (resp. smaller) than expected. Lucky agents (those that perform better in a given period) obtain more capital to manage in the next period, not because they are more skilled (performance is i.i.d. across agents and across periods) but because this provides incentives to report good performance instead of diverting output. In contrast with the symmetric information case, insurance is imperfect, because full insurance is not incentive compatible. So, the optimal mechanism exposes agents to a fraction of their idiosyncratic risk.

From a mathematical viewpoint, finding the optimal mechanism is challenging, as we need to extend to an infinite number of agents the martingale techniques introduced by Sannikov [46] in the one agent case. With only one agent, the Bellman equation that characterizes the optimal mechanism involves the partial derivatives of the value function with respect to two state variables: capital and the continuation utility promised to the (single) agent by the principal. In contrast, in our model with a continuous approximation of an economy with a large number of agents, the state variables are aggregate capital and the **entire distribution** of continuation utilities across agents, which belongs to the space of probability distributions over $\mathbb{R}$. Thus, we need to use mean-field games techniques,[3] because the value function of the principal solves a Bellman equation in an infinite dimensional space. We first determine this Bellman equation, which involves a generalized notion of derivative (the L-derivative[4]) of the value function with respect to the probability distribution of continuation utilities. Then, thanks to our log utility specification, we show that the dimension of state variables

[1] See also Rampini and Viswanathan [44].

[2] The principal in our model signs a dynamic contract with each agent. The set of contracts between the principal and the population of agents is the mechanism.

[3] Mean-field games techniques, introduced by Lasry and Lions [41]allow to approximate the solutions of games with a large number $N$ of agents by the solution of the limit game associated with $N = \infty$

[4] See Carmona and Delarue [13].

can be reduced to two: aggregate capital and the expectation of (a function of) agents' continuation utilities. These are sufficient statistics for the characterization of the optimal mechanism.[5] Thanks to the reduction in the dimension of the state space from infinity to two, we can fully characterize the dynamics of capital and consumption allocations as well as the distribution of continuation utilities across agents.

The optimal direct mechanism is remarkably simple: consumption and capital are allocated among agents proportionally to each agent's equivalent permanent consumption, defined as the constant lifetime stream of consumption giving the agent the same continuation utility as the mechanism. The equivalent permanent consumption of each agent grows at a constant rate in expectation, but is impacted by the agent's performance. The innovation in the growth rate of an agent's consumption or capital is proportional, by a positive constant $y$, to the agent's idiosyncratic output shock. The proportionality constant $y$ measures the extent to which the agent is exposed to the risk of his/her idiosyncratic output shock. Raising $y$ reduces allocative efficiency by reducing insurance, but it also relaxes the incentive compatibility condition, enabling the principal to extract more rents. Thus there is a rent-efficiency trade-off. We characterize the set of information-constrained Pareto optimal allocations, which can be parametrized by $y$. The larger the agents' initial outside utility, the larger the fraction of the surplus they must be given, the lower their risk exposure $y$. Because agents are exposed to their idiosyncratic shocks, inequality increases over time and agents become more and more heterogenous.[6] Moreover, while aggregate capital and output grow over time, growth is lower than under symmetric information. This is because incentive compatibility constrains how much new capital can be delegated to agents.

As an application, we study if the information-constrained optimal allocation obtaining in our framework can arise as an equilibrium allocation. To do so, we study equilibrium in a market in which agents exchange goods against money issued by the principal, and are subject to wealth taxes levied by the principal. When trading in the market, agents face a dynamic portfolio problem à la Merton [42]. They choose how much to invest in capital and in money, bearing in mind that the latter is exposed to inflation but the former is risky. The principal influences this portfolio choice by controlling money supply and thus the inflation rate. An appropriate monetary policy gives rise to an inflation rate such that, in equilibrium, the agents choose the same risk exposure in their portfolio as in the optimal mechanism.

Our results can be contrasted with the classical welfare theorems. These theorems state that, in a convex economy with complete markets and without frictions, all competitive equilibria are efficient (first welfare theorem) and, conversely, all efficient allocations can be decentralized by a competitive equilibrium after appropriate transfers between agents (second welfare theorem). The classical welfare theorems do not apply in our economy with asymmetric information and endogenously incomplete markets. Thus, in contrast with the first theorem of welfare, laissez faire competitive equilibria in our framework are generically constrained inefficient. However, all constrained optimal allocations can be implemented as market equilibria, provided there are appropriate taxes and an appropriate supply of money, which agents exchange against capital in order to buffer their productivity shocks. Our implementation result can thus be viewed as an extension of the second welfare theorem to an economy with endogenously incomplete markets.

[5] Angeletos [3] also avoids the "curse of dimensionality" with a log utility specification. A major difference is that in Angeletos [3] institutions and market incompleteness constraints are exogenous while in our paper, they are features of the endogenous optimal mechanism.

[6] More precisely, the coefficient of variation (standard deviation divided by the mean) of continuation utilities across agents increases over time.

**Literature:** Our paper complements several strands of literature.

First, our analysis of dynamic contracting between one principal and many agents is related to the literature analyzing dynamic contracting between one principal and one agent, in particular the seminal work of DeMarzo and Fishman [14], [15] and Sannikov [46], and the following analyses of Biais, Mariotti, Plantin, and Rochet [8], DeMarzo and Sannikov [16], Feng and Westerfield [23], and Di Tella and Sannikov [21]. As in Biais, Mariotti, Rochet, and Villeneuve [9] and DeMarzo, Fishman, He, and Wang [17], firm size is determined by the optimal contract and is useful to provide incentives. However, in contrast to these last two papers, in the present paper there are no capital adjustment costs. This enhances tractability, and gives rise to continuous reallocation of capital. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is output.

The major contribution of the present paper relative to that literature is to embed the contracting problem into a general equilibrium context, with a population of agents and aggregate resource constraints. Thus we shed light on the impact of incentive constraints on the allocation of capital and consumption between agents. In particular, we show that incentive constraints imply that inequality between agents increases over time. Moreover, we show how constrained optimal allocations can be implemented if the principal issues money which the agents can store or trade against goods and sets appropriate tax rates. We thus extend the second welfare theorem to an economy with frictions.

Second, our analysis is related to the dynamic macrofinance literature analyzing risk with exogenously incomplete markets (see Bewley [6], Aiyagari [1], Huggett [34] and [35], Krusell and Smith [39], Angeletos [3], He and Krishnamurthy [31] and [32], Brunnermeier and Sannikov [11], Di Tella [20], and Achdou et al [4]).[7]

The major contribution of the present paper relative to that literature is to provide microfoundations for market incompleteness.[8] Thus, the institutions and constraints we consider are endogenous features of the optimal dynamic mechanism. This helps clarify the consequences of informational frictions. For example, we reconcile two effects which, as explained by Angeletos [3], had so far been viewed as distinct. While the literature in line with Bernanke and Gertler (1989) emphasizes how wealth affects the *ability* to invest in capital, Angeletos [3] emphasizes how wealth affects the *willingness* to hold risky capital. Our mechanism design approach clarifies the common origin of these two forces: incentive compatibility constrains both how much capital agents are allocated and how much of the corresponding idiosyncratic risk they must bear. Consequently, in contrast with Angeletos [3], in our analysis frictions unambiguously lower capital accumulation. Another contribution of our analysis is that we prove that it is optimal to use the average of (a function of) agent's continuation utility as state variable, instead of the whole distribution of agents' continuation utilities. This provides an interesting complement to the analysis of Krusell and Smith [39], who show numerically that in their dynamic stochastic equilibrium the average of agents' wealths is almost perfectly in line with the behaviour of macroeconomic aggregates.

Building on the dynamic contracting framework developed in the present paper, which provides micro-foundations for endogenous market incompleteness and linear taxation, Gersbach, Rochet and von Thadden [24] extend the scope of investigation to the case in which there are two types of agents: entrepreneurs who run risky projects, and agents whose savings can be invested in corporate debt

[7] In particular, our focus on the distribution across agents and our reliance on mean-field techniques are in line with Achdou et al [4].

[8] Another difference is that, while most of that literature studies labor income risk, our paper, like Angeletos [3] considers capital return risk.

and public debt. Gersbach, Rochet and von Thadden [25] extend the scope of investigation further by interpreting the principal as a central bank and the agents as commercial banks. In that setting banks have a dual role as loan providers and money creators.

Third, our focus on money in the implementation of the optimal mechanism links our paper to the new monetarist literature initiated by the seminal papers of Kiyotaki and Wright ([36], [37]) and reviewed in Williamson and Wright [50]. A common theme with that literature is that money arises endogenously, as a useful instrument, instead of being a constraint as in cash in advance models or exogenous as in money-in-the-utility-function models. Money in our implementation encodes the memory of past performance in line with Kocherlakota [38] and provides consumption insurance in line with Berentsen and Rocheteau [5].

There are important differences, however, between our analysis and the new monetarist literature. First, instead of starting from a characterization of optimal allocations in a setting with money, we characterize the optimal mechanism in a real economy with only goods and no money, and then we introduce money as a tool to implement the optimal mechanism. Second, while the new monetarist literature assumes large households (Shi [47]) or the alternation of decentralized and centralized markets (Lagos and Wright, [40]) so that at the beginning of each period, all agents start with the same amount of money, in our framework, agents have endogenously heterogeneous money holdings, and we characterize the dynamics of this heterogeneity.[9] The third difference is a consequence of the second one: In the new monetarist literature, agents are homogenous at the beginning of each period, so the optimal allocation is pinned down by a static mechanism. In contrast, in our setting agents' continuation utilities vary stochastically over time, so the optimal allocation is set by a dynamic mechanism.[10]

Finally, we complement the mechanism design approach to optimal taxation pioneered by Mirrlees [43], Diamond and Mirrlees [19], and Diamond [18], and further developed by the new dynamic public finance literature, e.g., Golosov, Kocherlakota and Tsyvinski [26], Golosov and Tsyvinski [27], and Farhi and Werning [22]. A major difference is that, in these papers, risk and information asymmetry are about wage earners' skills, while, in our paper, risk and information asymmetry are about managers' capital returns. Correspondingly, unlike in these papers, the dynamic of capital allocation plays a key role in our analysis. Another major difference is that the optimal taxation literature focuses on one policy tool, namely the tax system, while in our set-up, the implementation of the optimal mechanism relies on money issuance as well as on taxation.

**Structure of the paper:** The rest of the paper is organized as follows. Section 2 introduces the continuous time model, and solves the symmetric information case, which provides a useful benchmark for the analysis of the asymmetric information case. In Section 3 we determine the Bellman equation that characterizes the principal value function under asymmetric information. Then we make a guess

[9]Rocheteau, Weill, and Wong [45] offer an interesting extension of the Lagos and Wright [40] approach in which they characterize the equilibrium distribution of money holdings. But, in line with previous new monetarist analyses, their approach relies on money from the start, in contrast with our approach, which starts with a direct mechanism without money and then implement it with money.

[10]Another interesting paper in line with that literature to which our paper is related is Aiyagari and Williamson [2]. In our paper like in theirs a continuum of agents have random outputs and want to share risk, but this is difficult because individual outputs are privately observed. A major difference between their work and ours is that in Aiyagari and Williamson [2] agents don't use capital, while in our analysis agents' ouptuts are increasing in the capital they are allocated. Thus, in our paper, unlike in Aiyagari and Williamson [2] capital allocation is key in the provision of incentives, and information asymmetry reduces capital accumulation relative to the first best. Another difference is that Aiyagari and Williamson [2] study the consequences of transportation constraints, which are absent in our framework.

on the form of the optimal policy, qualitatively close to that obtained under symmetric information, and finally we show that this candidate policy is indeed the full solution of our problem. Section 4 shows that the optimal direct mechanism can be implemented with money and taxes. Section 5 concludes. Proofs that are not in the main text are in Appendix A. Appendix B provides a brief introduction to generalized differential calculus in Wasserstein spaces.

# 2 The model

We model an economy with an infinite horizon and continuous time. Idiosyncratic shocks are captured by independent Brownian motions, which are easy to define when there is a finite number $N$ of agents, but more difficult with a continuum. We therefore start by describing the model with $N$ agents and then take the limit as $N$ tends to infinity.

## 2.1 The economy with finitely many agents

Agents, indexed by $i = 1, ...N$, and principal are infinitely lived with discount rate $\rho$ and logarithmic utility. There is a single good, which can be used for consumption or as capital input in a stochastic constant returns to scale technology operated by the agents. If agent $i$ invests $k_t^{(N),i}/N$ units[11] of the good in his production process, his instantaneous output (net of depreciation) is

$$dY_t^{(N),i} = \frac{k_t^{(N),i}}{N}[\mu dt + \sigma dZ_t^i], \tag{1}$$

where $\mu$ is the expected rate of return (net of depreciation) of the technology and $(Z_t^i)$, $i = 1, ...N$, are independent and identically distributed Brownian motions, whose increments can be interpreted as idiosyncratic non persistent productivity shocks. Let $(\mathcal{F}_t^{(N)})_{t\geq 0}$ be the filtration generated by the $N$-dimensional Brownian motion $(Z_t^1, \ldots, Z_t^N)_{t\geq 0}$. All processes introduced in this section are assumed to be square-integrable and adapted to $(\mathcal{F}_t^{(N)})$. The total amount of capital in the economy at time $t$ is[12]

$$K_t^{(N)} := \frac{1}{N}\sum_{i=1}^{N} k_t^{(N),i}. \tag{2}$$

If agent $i$'s instantaneous consumption at time $t$ is $c_t^{(N),i}/N$ and that of the principal $c_t^{(N),P}$, the law of motion of total capital is

$$dK_t^{(N)} = \left(\mu K_t^{(N)} - \frac{1}{N}\sum_{i=1}^{N} c_t^{(N),i} - c_t^{(N),P}\right) dt + \frac{\sigma}{N}\sum_{i=1}^{N} k_t^{(N),i} dZ_t^i. \tag{3}$$

Equation (3) is an intertemporal resource constraint stating that net aggregate investment is equal to total output (net of depreciation) minus total consumption.

[11]We denote capital and consumption allocations of each agent in this way (divided by $N$) to simplify the comparison with the limit case $N = \infty$. In some sense, each agent has "mass" $1/N$, so as to keep the total mass of agents to 1. In this way, aggregate capital and consumption converge to the integral of individual capital and consumption allocations when $N$ goes to $\infty$.

[12]Throughout the paper, individual and aggregate capital are strictly positive at every point in time. It is never optimal to run down the capital to zero in every case we are considering.

We seek to characterize the Pareto frontier of the economy by computing the value function $V$ of the principal, defined as the maximum expected utility she can obtain with a total volume of capital $K$ when agent $i$ is guaranteed a total expected utility $\omega^i$ for $i = 1, ...N$. This value function is obtained by finding capital and consumption paths $(k_t^{(N),i})$, $(c_t^{(N),i})$, and $(c_t^{(N),P})$ that maximize the principal's expected utility

$$\mathbb{E}\int_0^{\infty} e^{-\rho t}\log c_t^{(N),P}dt, \tag{4}$$

where $\rho > 0$ is the discount rate and maximization is subject to the law of motion of aggregate capital (3), the initial condition $K_0^{(N)} = K$, the agents' participation constraints:

$$\mathbb{E}\int_0^{\infty} e^{-\rho t}\log c_t^{(N),i}dt = \omega^i, \tag{5}$$

for $i = 1, ..., N$, and the capital allocation constraint:

$$K_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} k_t^{(N),i}.$$

In the above contracting problem we assume that agents and principal commit to the contract concluded at date 0.[13] Depending on the informational environment, there will be further constraints beyond those specified above. We first consider the case of symmetric information, in which idiosyncratic shocks and thus individual outputs are publicly observable. This case (the first best) will later serve as a benchmark for the case in which agents privately observe shocks and can secretly divert output.

## 2.2 Allocations under symmetric information with finitely many agents

In the first best, capital allocation has no impact on the dynamics of the continuation payoffs, and only matters for the volatility of aggregate capital. Risk aversion implies that this volatility should be minimized, and therefore that $k_t^{(N),i} \equiv K_t^{(N)}$. We can therefore drop the $(k_t^{(N),i})$ from the list of controls. Since production and capital accumulation are Markovian and preferences are stationary, there is no need to condition on previous realizations. Hence, we can assume that the controls $(c_t^{(N),i})$ and $(c_t^{(N),P})$ are feedback controls and thus only depend on the current state $K_t^{(N)}$, the current time $t$, and the initial values $\omega^i$, $1 \leq i \leq N$. More precisely, we assume that there are functions $\bar{c}^A$ and $c^P$ such that

$$c_t^{(N),i} = \bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) \text{ and } c_t^{(N),P} = c^P(t, K_t^{(N)}, \nu^{(N)}),$$

where $\nu^{(N)}$ denotes the empirical measure of the $N$ expected utilities at time 0. Since all agents are allocated the same amount of capital, the law of motion of aggregate capital (3) becomes

$$dK_t^{(N)} = \left(\mu K_t^{(N)} - \frac{1}{N}\sum_{i=1}^{N}\bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) - c^P(t, K_t^{(N)}, \nu^{(N)})\right)dt + \sigma K_t^{(N)} dA_t^{(N)}, \tag{6}$$

where

$$A_t^{(N)} \equiv \frac{1}{N}\sum_{i=1}^{N} Z_t^i \tag{7}$$

[13] As a consequence, we do not specify outside options for the agents beyond time 0.

are normal random variables with mean 0 and variance $\frac{t}{N}$. The principal's problem therefore is to determine feedback controls $\bar{c}^A, c^P$ that maximize (4) subject to the law of motion (6), the initial condition $K_0^{(N)} = K$, and the agents' participation constraints (5). We now simplify the problem by looking at the limit case where $N \to \infty$.

### 2.3 Optimal allocations under symmetric information in the limit economy

We assume that $\nu^{(N)}$ weakly converges to some probability $\mathbb{P}$ which represents the initial distribution of utilities. When $N \to \infty$, the propagation of chaos theory (Sznitman, [48]) shows that the agents' average consumption

$$\frac{1}{N}\sum_{i=1}^{N} \bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) = \int \bar{c}^A(t, K_t^{(N)}, \omega, \nu^{(N)}) d\nu^{(N)}(\omega)$$

converges to

$$c^A(t, K_t, \mathbb{P}) = \int_{\mathbb{R}} \bar{c}^A(t, K_t, \omega, \mathbb{P})\, d\mathbb{P}(\omega) \tag{8}$$

and the stochastic differential equation (6) converges to the ordinary differential equation

$$\dot{K}_t = \mu K_t - c^A(t, K_t, \mathbb{P}) - c^P(t, K_t, \mathbb{P}). \tag{9}$$

When $N$ goes to infinity, there is no risk any more. Intuitively, this is because, by the law of large numbers, $A_t^{(N)}$ goes to 0 as $N$ goes to infinity. Thus capital accumulation and consumption are deterministic.[14] These remarks drastically simplify the control problem, which can now be written as

$$\max_{\bar{c}^A, c^P} \int_0^\infty e^{-\rho t} \log c^P(t, K_t, \mathbb{P}) dt \tag{10}$$

subject to the law of motion (9), the initial condition $K_0 = K$, the agents' participation constraints

$$\int_0^\infty e^{-\rho t} \log \bar{c}^A(t, K_t, \omega, \mathbb{P}) dt = \omega \tag{11}$$

for all $\omega \in \mathbb{R}$, and the definition of agents' aggregate consumption (8). Since optimality implies the transversality condition $\lim_{t\to\infty} e^{-\mu t} K_t = 0$, (9) and the initial condition $K_0 = K$ together can be integrated into:

$$K = \int_0^\infty e^{-\mu t} \left( c^A(t, K_t, \mathbb{P}) + c^P(t, K_t, \mathbb{P}) \right) dt. \tag{12}$$

Taking first-order conditions of the above defined problem, we obtain our first lemma (whose proof is in Appendix A):

**Lemma 1.** *The solution of (10) subject to (11) and (12) (if it exists), is such that aggregate consumption is a constant fraction of capital, equal to their discount rate $\rho$:*

$$c^P(t, K_t, \mathbb{P}) + c^A(t, K_t, \mathbb{P}) = \rho K_t, \tag{13}$$

[14] To facilitate the exposition, the argument is only made heuristically at this stage. In Section 3 we develop this theory in full rigour, assuming Lipschitz continuity of the controls with respect to the measure $\mathbb{P}$.

*and the consumption of the principal is a constant fraction of capital:*

$$c^P(t, K_t, \mathbb{P}) = \gamma^P K_t = (\rho - ae^{-\frac{\mu-\rho}{\rho}})K_t, \tag{14}$$

*where*

$$a \equiv \frac{1}{K}\int_{\mathbb{R}} e^{\rho\omega} d\mathbb{P}(\omega). \tag{15}$$

The parameter $a$ relates the initial values of the two state variables $K$ and $\mathbb{P}$ of the principal's problem to each other. For each outside option $\omega$, $e^{\rho\omega}$ can be interpreted as an agent's "equivalent permanent consumption"for $\omega$, namely the constant lifetime stream of consumption that would give the agent total utility $\omega$. Integrating over all agents with respect to distribution $\mathbb{P}$, this gives the total expected equivalent permanent consumption demanded by all agents. This is set in relation to $K$, the total capital stock at the disposal of the principal to satisfy these demands over time.

Since the principal's consumption must be positive, (14) implies that the problem has a solution only if

$$a < \rho e^{\frac{\mu-\rho}{\rho}}$$

i.e., if the following condition on the initial data $K, \mathbb{P}$ of the principal's problem holds:

$$\rho K e^{\frac{\mu-\rho}{\rho}} > \int_{\mathbb{R}} e^{\rho\omega} d\mathbb{P}(\omega). \tag{16}$$

Intuitively, the condition means that the initial amount of aggregate capital ($K$) is sufficiently large to give his/her reservation utility ($\omega$) to each agent, while leaving enough resources available for the principal to have non-negative consumption.

Equation (13) states that aggregate consumption is a constant fraction $\rho$ of aggregate capital $K$. This stems from our assumption that the principal and the agents have logarithmic utility. For the same reason, the consumption of the principal is a constant fraction of aggregate capital, as stated in Equation (14). In turn, this implies that the aggregate consumption of the agents is a constant fraction of aggregate capital. Our next proposition spells out how this aggregate consumption is allocated across individual agents.

To conclude the characterization of the optimal allocation under symmetric information, we define the expected utility at any time $t$ of an agent who starts out with reservation utility $\omega$ as

$$\omega_t = \int_t^{+\infty} e^{-\rho(s-t)} \log \bar{c}^A(s, K_s, \omega, \mathbb{P}) ds. \tag{17}$$

Introducing the new variable $\omega_t$ makes it possible to replace the integral constraints (11) by the initial conditions $\omega_0 = \omega$ and work recursively. Differentiating (17) yields the ODE

$$\dot{\omega}_t = \rho\omega_t - \log \bar{c}^A(t, K_t, \omega, \mathbb{P}), \quad \text{with} \quad \omega_0 = \omega. \tag{18}$$

The next proposition summarizes the properties of the optimal allocation under symmetric information.

**Proposition 1.** *The first-best contracting problem has a solution if and only* $a < \rho e^{\frac{\mu-\rho}{\rho}}$. *In this case we have the following properties:*

1. *The value function is*
$$V(K,\mathbb{P}) = \frac{1}{\rho}\left(\log K + \log(\rho e^{\frac{\mu-\rho}{\rho}} - a)\right). \tag{19}$$

2. *Capital grows at the constant rate $\mu - \rho$:*
$$K_t = K e^{(\mu-\rho)t}. \tag{20}$$

3. *Agents' continuation utilities grow linearly:*
$$\omega_t = \omega + \frac{\mu-\rho}{\rho} t. \tag{21}$$

4. *At each date $t$, an agent with continuation utility $\omega_t$ consumes $\gamma^A e^{\rho\omega_t}$, where $\gamma^A = e^{-\frac{\mu-\rho}{\rho}}$.*

5. *At each date $t$, the principal consumes a constant fraction $\gamma^P = \rho - \gamma^A a$ of aggregate capital $K_t$.*

Using (19) and the definition of $a$, we have that
$$e^{\rho V} + \int e^{\rho\omega} d\mathbb{P} = \rho K e^{\frac{\mu-\rho}{\rho}}, \tag{22}$$
The left-hand side of (22) is the sum of the principal's equivalent permanent consumption and the aggregate agents' permanent consumption, while the right-hand side is the total amount of equivalent permanent consumption to be allocated among the principal and the agents. Thus, (22) describes the Pareto frontier in the space of equivalent permanent consumptions, pinning down how total surplus is divided among agents and principal.

Equation (13) stated that aggregate consumption was a constant fraction $\rho$ of aggregate capital $K$. Since capital yields output at rate $\mu$, the investment rate is $\mu - \rho$, which gives rise to the dynamics of the capital stock stated in (20).

Equation (14) stated that the consumption of the principal was a constant fraction $\gamma^P$. Jointly with (13) and the dynamics of agent's utility (21) this yields the value of $\gamma^A$ in Proposition 1.

# 3 Optimal allocations under asymmetric information

We now turn to the case in which agents privately observe their individual output. By the revelation principle, we consider direct revelation mechanisms that are incentive compatible. A mechanism is a mapping from the output $\hat{Y}_t^i$, reported and delivered by agent $i$ to the principal, into consumption $(c_t^i)_{t\geq 0}$ and capital allocations $(k_t^i)_{t\geq 0}$ for the agent. The processes $(c_t^i)_{t\geq 0}$ and $(k_t^i)_{t\geq 0}$ are positive and adapted to the filtration $(\mathcal{F}_t)$ generated by the $N$ reported outputs $\hat{Y}_t^i$. Since agents privately observe their own output, they can be tempted to divert a part of it and secretly consume it. To avoid this, the mechanism must induce truthful revelation, i.e., it must be incentive compatible.

## 3.1 Incentive compatibility

Consider an agent with contract $(c_t^i, k_t^i)_{t\geq 0}$ who considers under-reporting his Brownian increments by some amount $\Delta_t dt$. Defining
$$d\hat{Z}_t^i = dZ_t^i - \Delta_t\, dt, \tag{23}$$

the dynamics of reported output writes as

$$d\hat{Y}_t^i = \mu k_t^i \, dt + \sigma k_t^i d\hat{Z}_t^i.$$

Since the agent cannot secretly store, diversion cannot be negative: $\Delta_t \geq 0$ for every $t$. The time t expected utility of agent $i$ who adopts a diversion strategy $(\Delta_s)_{s \geq t}$ is

$$\omega_t^{i,\Delta} = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \log(c_s^i + \sigma k_s^i \Delta_s) \, ds \right].$$

The martingale representation theorem implies that the dynamics of the agent's continuation utility is

$$d\omega_t^{i,\Delta} = (\rho \omega_t^{i,\Delta} - \log(c_t^i)) \, dt + \sigma y_t^i d\hat{Z}_t^i, \tag{24}$$

where $y_t^i$ is a $\mathcal{F}_t$-adapted process. The drift term on the right-hand side of (24) is similar to that prevailing in the first best, see (18). The second term on the right-hand side of (24) is the product of the agent's reported productivity shock ($d\hat{Z}_t^i$) by $\sigma y_t^i$. Intuitively, $y_t^i$ is the sensitivity of the agent's continuation utility with respect to his/her report.

To provide incentives for truthful revelation, the principal must propose a contract generating an appropriate sensitivity. We now offer an intuitive examination of the corresponding incentive compatibility condition. Equation ( 24) implies the local incentive compatibility condition is that the agent $i$ be better off revealing $dZ_t^i$ truthfully, and getting

$$\log(c_t^i)dt + \sigma y_t^i dZ_t^i$$

than under-reporting $d\hat{Z}_t^i = dZ_t^i - \Delta dt$ and getting

$$\log(c_t^i + \sigma k_t^i \Delta)dt + \sigma y_t^i d\hat{Z}_t^i = \log(c_t^i + \sigma k_t^i \Delta)dt + \sigma y_t^i (dZ_t^i - \Delta dt).$$

Therefore, the local incentive compatibility condition is

$$\sigma y_t^i \geq \sup_{\Delta \geq 0} \frac{\log((c_t^i + \sigma k_t^i \Delta) - \log(c_t^i)}{\Delta} = \frac{\sigma k_t^i}{c_t^i}.$$

This means that, for incentive compatibility, the sensitivity of continuation utility to performance must be larger than the product of the capital $k_t^i$ managed by the agent by his/her marginal utility of consumption, i.e.,

$$y_t^i \geq \frac{k_t^i}{c_t^i}.$$

The following lemma, whose proof is in Appendix A, states that the condition above is also sufficient.

**Lemma 2.** *The incentive compatibility condition is equivalent to the inequality*

$$\forall t, \quad y_t^i \geq \frac{k_t^i}{c_t^i}. \tag{25}$$

The incentive compatibility condition (25) implies that, in contrast with the symmetric information case, agents cannot fully share the risk of their idiosyncratic shocks. Condition (25) also shows there is a tradeoff between risk-sharing and investment: providing more insurance to the agent, by reducing the sensitivity of his/her continuation value to output shocks is possible only at the cost of reducing capital relative to consumption. This is because increasing capital, and therefore output, increases the amount of resources the agent can divert, which tightens the incentive constraint. This tradeoff is similar to that arising in Biais, Mariotti, Rochet and Villeneuve [9], where the size of operation (similar to capital in the present context) was limited by incentive compatibility.

### 3.2 Mean-field economy

As in the symmetric information, to simplify the problem we consider the limiting case in which $N$ tends to infinity. As in the first best, the value function of the principal will depend on the capital stock $K$ and the distribution of agents' continuation payoffs $\mathbb{P}$. We introduce the space $\mathcal{P}_2(\mathbb{R})$ of probability measures on $\mathbb{R}$ with a finite second moment, which we endow with the Wasserstein distance $W_2$ (see for example Sznitman [48], Carmona and Delarue [13], Cardaliaguet [12] and Villani [49] and Appendix B). The main difference with the first best case is that the principal has to leave a part of the idiosyncratic risk to each agent, for incentive reasons. The agents being risk averse, it is never optimal for the principal to expose them to more risk than required by the incentive compatibility condition. That is, in any optimal allocation, the incentive constraint (25) is always binding and we can eliminate the capital allocation variable by writing $k_t^i = y_t^i c_t^i$. As a consequence, the individual continuation utility of agent $i$ will evolve in the $N$ agents economy as

$$d\omega_t^{i,(N)} = (\rho\omega_t^{i,(N)} - \log(c_t^i))\, dt + \sigma \frac{k_t^i}{c_t^i} dZ_t^i.$$

Moreover, we denote by $\mathbb{P}_t^{(N)} = \frac{1}{N}\sum_{i=1}^N \delta_{\omega_t^{i,(N)}}$ the empirical measure, where $\delta_x$ denotes the Dirac measure in $x$. We now assume the following feedback nature of the controls:

$$c_t^i = c(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \tag{26}$$
$$k_t^i = k(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \tag{27}$$
$$c_t^P = c^P(K_t^{(N)}, \mathbb{P}_t^{(N)}) \tag{28}$$

for each $i = 1, ...N$, where $c$, $c^P$, and $k$ are Lipschitz functions $\mathbb{R}^+ \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^+ \to \mathbb{R}^+$. More precisely, there exists a constant $L > 0$, such that for $f \in \{c, k, c^P\}$,

$$|f(K_1, \mathbb{P}_1, \omega_1) - f(K_0, \mathbb{P}_0, \omega_0)| \leq L(|K_1 - K_0| + |\omega_1 - \omega_0| + W_2(\mathbb{P}_1, \mathbb{P}_0)). \tag{29}$$

The theory of propagation of chaos (see Sznitman[48] section 1 for details) shows that, when $N \to \infty$, for any fixed integer $m$, the joint distribution of the $m$+1-dimensional process $(K_t^{(N)}, \omega_t^{1,(N)}, \ldots, \omega_t^{m,(N)})_{t\geq 0}$ weakly converges to the product distribution $\delta_{K_t} \otimes \mathbb{P}_t^{\otimes^m}$ where

$$dK_t = \left(\mu \int k(K_t, \mathbb{P}_t, .) d\mathbb{P}_t - \int c(K_t, \mathbb{P}_t, .) d\mathbb{P}_t - c^P(K_t, \mathbb{P}_t)\right) dt, \quad K_0 = K, \tag{30}$$

and $\mathbb{P}_t$ is the marginal distribution of the mean-field stochastic differential equation (SDE)

$$d\omega_t = [\rho\omega_t - \log c(K_t, \mathbb{P}_t, \omega_t)]dt + \sigma y(K_t, \mathbb{P}_t, \omega_t) dZ_t, \tag{31}$$

where $(Z_t)_t$ is a Brownian motion and $y$ denotes the ratio $\frac{k}{c}$.
As can be seen in equation (30), when $N \to \infty$, the dynamics of aggregate capital is deterministic. Intuitively, as in the symmetric information case, the idiosyncratic shocks average out according to a form of law of large numbers.
Under the Lipschitz condition (29) and given an initial condition, there exists a unique solution $(K_t, \mathbb{P}_t)$ to the above mean-field SDE. Hereafter, we focus on the limit case in which $N$ goes to infinity. The pair $(K_t, \mathbb{P}_t)$ plays the role of the state variable of the principal problem in the mean-field large economy on which we apply the recursive formulation while the set of admissible controls is the set of feedback controls satisfying the Lipschitz condition (29).

### 3.3 Characterization of second best allocations.

The main difficulty for exploiting the dynamic programming principle is to differentiate functionals defined on the Wasserstein space. Various notions of derivatives with respect to measures have been developed, in connection with the theory of optimal transport[15]. We use here the Wasserstein metric on the space of probability measures and the notion of L-differentiability that is presented in Appendix B. Following the traditional approach for control problems, we first determine the shape of the Hamilton-Jacobi-Bellman (HJB) equation that the value function of the principal must satisfy (necessary condition) and then establish a verification theorem showing that regular solutions of this HJB equation solve our control problem (sufficient condition). To do so, consider the control problem of the principal

$$V(K, \mathbb{P}) = \sup_{(c, c^P, y) \in \mathcal{K}} \int_0^\infty e^{-\rho t} \log c_t^P dt, \tag{32}$$

where the state equations are given by the capital allocation constraint (30) and the stochastic differential equation (31). The supremum is taken over the set $\mathcal{K}$ of admissible feedback controls $(c, c^P, y)$, such that for all $t \geq 0$,

$$\int y(., \omega) c(., \omega) d\mathbb{P}_t(\omega) = K_t. \tag{33}$$

Since the process $K_t$ is deterministic, in any optimal solution $c_t^P$ has to be deterministic, which simplifies the formulation of the problem.

A second difficulty is that this control problem involves a constraint (33) that mixes control variables and state variables. To deal with this constraint, we introduce a related, unconstrained, problem as follows: for each function $\lambda$ (defined on the product space $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$), which we will call from now on the *Lagrange multiplier*, consider the control problem

$$V_\lambda = \sup_{(c, c^P, y)} \int_0^\infty e^{-\rho t} \left( \log c^P(K_t, \mathbb{P}_t) + \lambda(K_t, \mathbb{P}_t) \left( K_t - \int y(., \omega) c(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt.$$

We first state a result that establishes a link between the principal's value $V$ and $V_\lambda$.

**Proposition 2.** *If i) for every Lagrange multiplier process one can find an optimal control* $\alpha_\lambda = (c_\lambda, c_\lambda^P, y_\lambda)$ *such that*

$$V_\lambda = \int_0^\infty e^{-\rho t} \left( \log c_\lambda^P + \lambda(K_t, \mathbb{P}_t) \left( K_t - \int y_\lambda(., \omega) c_\lambda(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt$$

*and ii) there exists a Lagrange multiplier* $\lambda_0$ *such that for all* $t \geq 0$,

$$K_t = \int y_{\lambda_0}(., \omega) c_{\lambda_0}(., \omega) d\mathbb{P}_t(\omega),$$

*i.e.* $\alpha_{\lambda_0} \in \mathcal{K}$, *then* $V = V_{\lambda_0}$ *and* $\alpha_{\lambda_0}$ *solves the principal problem.*

We are now in a position to derive the HJB equation associated with the unconstrained problem.

[15]See the books by Carmona and Delarue [13] and Villani [49] and our more formal presentation in Appendix B of the present paper.

**Proposition 3.** *If the value function of the principal $V$ is sufficiently regular,*[16] *it satisfies the following HJB equation:*

$$\rho V(K,\mathbb{P}) = \sup_{c,c^P,y}\Bigg\{\log c^P(K,\mathbb{P}) + \lambda_0(K,\mathbb{P})\left(K - \int y(K,\mathbb{P},\omega)c(K,\mathbb{P},\omega)d\mathbb{P}(\omega)\right) \tag{34}$$
$$+\frac{\partial V}{\partial K}(K,\mathbb{P})\left(\mu K - c^P(K,\mathbb{P}) - \int c(K,\mathbb{P},\omega)d\mathbb{P}(\omega)\right)$$
$$+\int \partial_\omega \nabla V[K,\mathbb{P}](\omega)(\rho\omega - \log c(K,\mathbb{P},\omega))d\mathbb{P}(\omega) + \int \partial_{\omega\omega}\nabla V[K,\mathbb{P}](\omega)\frac{\sigma^2}{2}y^2(K,\mathbb{P},\omega)d\mathbb{P}(\omega)\Bigg\},$$

*where $\nabla V$ denotes the L-gradient of $V$ with respect to the measure $\mathbb{P}$ and $\partial_\omega$ (respectively $\partial_{\omega\omega}$) denote its first (respectively second) partial derivative in $\omega$, while $\lambda_0$ denotes the Lagrange multiplier associated with the capital allocation constraint given in Proposition 2 ii).*

The first two lines of equation (34) are similar to their counterparts in the one-agent problem. The first term on the right-hand side represents the instantaneous utility from consumption, the second term reflects the capital allocation constraint, and the third term is the derivative of the value function with respect to capital multiplied by the growth rate of capital. The last two terms of equation (34) are similar to those that would arise in a one-agent problem in that they involve the first derivative of the value function of the principal with respect to the continuation utility of the agent multiplied by the drift of this continuation utility, and the second derivative multiplied a quadratic term arising because of Ito's lemma. They, however, differ from their one-agent counterpart because they reflect that the principal faces a population of agents, with different $\omega$, so that they involves integrals and L-gradients with respect to the measure $\mathbb{P}(\omega)$.

Inspired by classical verification theorems for stochastic control of diffusion processes, we prove in Appendix A the following result, which is a consequence of the Itô formula given in Appendix B for functions defined on the Wasserstein space.

**Proposition 4.** *(Verification Theorem) Let $\lambda(.)$ be a Lagrange multiplier, and $v^\lambda(K,\mathbb{P})$ be a regular function. If there exist i) a solution $v^\lambda$ to (34) with the transversality condition $\lim_{t\to+\infty} e^{-\rho t}v^\lambda(K_t,\mathbb{P}_t) = 0$, and ii) a control $\alpha^*_\lambda$ attaining the maximum in (34), then $v^\lambda = V_\lambda$. Moreover, if there is a Lagrange multiplier $\lambda_0$ such that $\alpha^*_{\lambda_0} \in \mathcal{K}$ then $v^{\lambda_0} = V$.*

## 3.4 A guess-and-verify approach

We now make a conjecture, a "guess", about the form of the solution to the optimal control problem and show that the corresponding value function satisfies the Hamilton-Jacobi-Bellman equation (34), so that the guess is the actual solution of the problem, by the verification theorem (Proposition 4).

### 3.4.1 The guess

In line with the first best case, we make the following conjecture

$$c^P(K,\mathbb{P}) = \gamma^P K,\ \ c(K,\mathbb{P};\omega_t) = \gamma^A \exp(\rho\omega_t),$$

[16] By this we mean that it is differentiable in $K$, L-differentiable in $\mathbb{P}$ (see Appendix B) and that its L-gradient in $\mathbb{P}$ is twice differentiable with respect to $\omega$.

where $\gamma^P$ and $\gamma^A$ are positive constants. We also guess that $y(.)$ is a constant, which we denote by $y$. The binding incentive compatibility constraint then implies that an agent's consumption is proportional to this agent's capital:

$$c(K, \mathbb{P}; \omega_t) = \frac{k(K, \mathbb{P}; \omega_t)}{y}.$$

Our conjecture implies that total consumption at date $t$ equals $(\gamma^P + \frac{1}{y})K_t$ and thus that $K_t$ grows at a constant rate

$$\frac{\dot{K}_t}{K_t} \equiv g = \mu - \gamma^P - \frac{1}{y}. \tag{35}$$

We now characterize the solution of the problem under our conjecture. Then we will show that this is also the solution to the full problem.

The principal's utility $V(K, \mathbb{P})$ under our conjecture satisfies

$$\rho V(K, \mathbb{P}) = \int_0^\infty \rho e^{-\rho t} \log(\gamma^P K \exp(gt))dt = \log(\gamma^p K) + \frac{g}{\rho}.$$

Similarly, by the participation constraint, an agent's initial utility $\omega$ satisfies

$$\rho\omega = \mathbb{E}\left[\int_0^\infty \rho e^{-\rho t} \log(\frac{k(K, \mathbb{P}; \omega_t)}{y})dt\right].$$

Integrating by parts, this becomes

$$\rho\omega = \mathbb{E}\left[\log \frac{k(K, \mathbb{P}, \omega)}{y} + \int_0^\infty \rho e^{-\rho t} d[\log k(K, \mathbb{P}; \omega_t)]\right].$$

It remains to determine the dynamics of $\log k(K, \mathbb{P}; \omega_t)$. We know that $k_t \equiv k(K, \mathbb{P}; \omega_t)$ is proportional to $exp(\rho\omega_t)$ and grows on average at rate $g$. Moreover by definition of $y$, the volatility of $\omega_t$ is $\sigma y$. Therefore, by Ito's lemma:

$$\frac{dk_t}{k_t} = \frac{d(exp(\rho\omega_t))}{exp(\rho\omega_t)} = gdt + \rho\sigma y dZ_t.$$

and:

$$d(\log k_t) = (g - \frac{\rho^2\sigma^2 y^2}{2})dt + \rho\sigma y dZ_t.$$

Consequently:

$$\rho\omega_t = \log \frac{k(K, \mathbb{P}; \omega_t)}{y} + \frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Hence the capital managed by an agent is proportional to this agent's equivalent permanent consumption $exp(\rho\omega_t)$:

$$exp(\rho\omega_t) = \frac{k(K, \mathbb{P}; \omega_t)}{y} exp\left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right]. \tag{36}$$

Integrating (36) with respect to the measure $\mathbb{P}$, the capital allocation constraint rewrites:

$$\int exp(\rho\omega_t)d\mathbb{P} = \frac{K_t}{y} exp\left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right]. \tag{37}$$

Equation (37) shows that, under our conjecture, the ratio of aggregate capital to expected equivalent permanent consumption is constant over time. More precisely, we have

$$a = \frac{\int exp(\rho\omega_t)d\mathbb{P}}{K_t} = \frac{1}{y}exp\left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right], \tag{38}$$

where the constant $a$ is as defined in the analysis of the symmetric information case in equation (15).

The value function of the principal is thus $V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(a)$ where

$$\rho v(a) = \max \log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} \tag{39}$$

under constraint (37). A priori, the value function $V$ depends on the aggregate capital $K$ and the entire distribution of continuation utilities $\mathbb{P}$. Inspecting (37) and (39), however, one can note that the value function and the constraint depend on the individual continuation utilities only through their expectation: $\int exp(\rho\omega)d\mathbb{P}$, which is therefore a sufficient statistics for $\mathbb{P}$. This implies that $V$ only depends on two scalar variables, $K$ and $a = \frac{\int exp(\rho\omega)d\mathbb{P}}{K}$. Expressing the constraint (37) in logs, the function $v$ is

$$\rho V(K, \mathbb{P}) = \log K + \rho v(a) = \log K + \sup_{\gamma^P, y} \log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho},$$

subject to the capital allocation constraint

$$\log(ya) = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Inserting the value of the growth rate from (35) into the capital allocation constraint (37) expressed as an inequality yields

$$a \leq \frac{1}{y}exp\left[\frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right].$$

This is consistent with $\gamma^P > 0$ only if $a < \frac{1}{y}exp[\frac{\mu - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}]$. Thus, like in the symmetric information case, the principal's problem only has a solution when $a$ is less than some value defined here as

$$a_{max} = \max_y \frac{1}{y}exp\left(\frac{\mu - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right).$$

Building on the above analysis, we obtain the next proposition, whose proof is in Appendix A:

**Proposition 5.** *Under our conjecture, the value function of the principal's problem is* $\frac{\log K}{\rho} + v(a)$, *where* $v(a)$ *is defined for* $0 < a < a_{max}$ *by*

$$\rho v(a) = \sup_{\gamma^P, y}\left(\log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho}\right), \tag{40}$$

$$s.t. \log ya = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

*The solution of this problem is such that* $\gamma^P = \rho - \frac{1}{y + \rho\sigma^2 y^3}$, *where* $y = y(a)$ *is defined implicitly by*

$$m = \frac{1}{y}exp[\frac{\mu - \rho}{\rho} - \frac{\rho\sigma^2 y}{1 + \rho\sigma^2 y^2} - \frac{\rho\sigma^2 y^2}{2}]. \tag{41}$$

The allocation of surplus between the principal and the agents is parametrized by $y$. As can be seen in (40) the utility of the principal increases with $y$. $a$ is the initial aggregate utility of the agents (per unit of initial capital), which they must be offered because of the participation constraint. The right-hand side of (41) is decreasing in $y$. So $y$ decreases with $a$. Thus, as $a$ goes up, the participation constraint of the agents gets more demanding, resulting in a lower value of $y$, corresponding to a lower utility for the principal.

### 3.4.2 Verifying our guess

We now show that our guess is correct: the value function $V(K,\mathbb{P}) = \frac{\log K}{\rho} + v(a)$ obtained under our guess satisfies the Bellman equation (34). To do so, we first compute the derivatives of $V$, using the definition $a = \frac{\int exp(\rho\omega)d\mathbb{P}}{K}$:

$$V_K = \frac{1}{\rho K} - \frac{av'(a)}{K},\ \nabla V = \exp(\rho\omega)\frac{v'(a)}{K}.$$

Then

$$\partial_\omega \nabla V = \rho \nabla V, \partial_{\omega\omega}\nabla V = \rho^2 \nabla V.$$

Substituting these derivatives, and allowing $\gamma^A$ and $y$ to depend on $\omega$ unlike in our simplifying guess (and correspondingly expressing an agent's consumption as $\gamma^A(\omega)exp(\rho\omega)$), the Bellman equation writes as

$$\rho v(K,a) = \sup_{y(.),\gamma^P,\gamma^A(.)} \Big[\log \gamma^P K + \lambda(K - \int y(\omega)\gamma^A(\omega)exp(\rho\omega)d\mathbb{P}) \tag{42}$$

$$+(\frac{1}{\rho K} - \frac{av'(a)}{K})[\mu K - \gamma^P K - \int \gamma^A(\omega)\exp(\rho\omega)d\mathbb{P}] + \int \rho exp(\rho\omega)\frac{v'(a)}{K}[-log\gamma^A(\omega) + \frac{\rho\sigma^2}{2}y^2(\omega)]d\mathbb{P}(\omega)\Big].$$

Pointwise maximization inside the integral shows that the optimal controls $(y(.),\gamma^A(.))$ are constant since $\omega$ only appears through the common factor $exp(\rho\omega)d\mathbb{P}(\omega)$. With constant controls $y$ and $\gamma^A$, the HJB equation takes a simpler expression

$$\rho v(K,a) = \sup_{y,\gamma^P,\gamma^A} \Big(\log \gamma^P K + \lambda(1 - y\gamma^A a)K$$
$$+(\frac{1}{\rho} - av'(a))[\mu - \gamma^P - \gamma^A a] + \rho a v'(a)\left[-log\gamma^A + \frac{\rho\sigma^2}{2}y^2\right]\Big).$$

Replacing $\gamma^A$ by $\frac{1}{ya}$, the second term cancels, reflecting that the capital allocation constraint holds as an equality. The Bellman equation becomes:

$$\rho v(K,a) = \log K + \sup_{y,\gamma^P}\left[\log\gamma^P + (1-\rho a v'(a))\frac{\mu - \gamma^P - \frac{1}{y}}{\rho} + \rho a v'(a)\left[\log(ya) + \frac{\rho\sigma^2 y^2}{2}\right]\right].$$

This coincides with the definition of $v$ given in proposition (5), once we have noted that the Lagrange multiplier of the capital alocation consraint equals $\rho a v'(a)$. Thus we can state the main result of our paper:

**Proposition 6.** *The value function of the principal's problem under asymmetric information is*

$$V(K,\mathbb{P}) = \frac{\log K}{\rho} + v(a),$$

*where the function $v$ is defined in Proposition (5).*

## 3.5 Properties of second best allocations

Taking stock of the analysis above, the next proposition summarizes the properties of optimal information-constrained allocations. These properties are drastically simplified by the fact that date $t$ allocations only depend on two scalars, namely the capital stock $K_t$ and the expected equivalent permanent consumption of agents $\int \exp(\rho\omega_t)d\mathbb{P}$, and that the ratio of the latter to the former is equal to the constant denoted by $a$, so that optimal controls can all be expressed as functions of $y = y(a)$ defined in Proposition 5.

**Proposition 7.** *Second best optimal allocations can be parameterized by $y$. They are such that, at each date $t$:*

1. *Aggregate capital grows at a constant rate $g = \mu - \rho - \frac{\rho\sigma^2 y}{1+\rho\sigma^2 y^2}$.*
2. *Agents' continuation utilities follow a drifted Brownian motion:*
$$\omega_t = \omega + \left(\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right)t + \sigma y Z_t. \tag{43}$$
3. *The capital managed by an agent is $k_t = \frac{exp(\rho\omega_t)}{a}$ and his consumption $c_t = \frac{k_t}{y}$.*
4. *The principal consumes a constant fraction of aggregate capital: $c_t^P = \gamma^P K_t$, where*
$$\gamma^P = \rho - \frac{1}{y + \rho\sigma^2 y^3}. \tag{44}$$
5. *The parameter $y$ can take any value in $(y_{min}, \infty)$, where $y_{min}$ is the unique positive root of the equation $\rho\sigma^2 y^3 + y = \frac{1}{\rho}$. This guarantees that the consumption of the principal is positive, i.e., $\gamma^P > 0$*

Property 1 shows that frictions reduce growth, since $g = \mu - \rho - \frac{\rho\sigma^2 y}{1+\rho\sigma^2 y^2}$ is lower than the first best growth rate $\mu - \rho$. This reflects incentive constraints, which restrict investment. When $\sigma = 0$, there is no incentive problem and the growth rate is equal to its first best level.

Property 2 implies that the cross section of agents' continuation payoffs gets more dispersed as time goes by. Even if all agents are ex-ante identical, inequality necessarily increases over time, due to incentive compatibility constraints. Moreover, there is a simple relation between the continuation utility of an agent at date $t$ and its performance over $(0, t)$. Indeed, the average productivity of the agent over $(0, t)$ is just $\mu + \sigma\frac{Z_t}{t}$. Optimal compensation implies a simple, affine, relation between the continuation utility $\omega_t$ and this performance measure, similarly to Holmstrom and Milgrom [33].

Properties 3 and 4 are similar to the first best case, in that consumptions are equal to capital multiplied by a constant. This simplicity is due to our assumption that utilities are logarithmic.

Property 5 states that information-constrained allocations are parameterized by the sensitivity of agent's continuation utility to performance, $y$ which can take any value such that $\sigma^2\rho^2 y^3 + \rho y > 1$, to ensure that $\gamma^P > 0$. This provides a simple description of the information constrained Pareto frontier which we present next.

### 3.6 Information constrained Pareto frontier

The above analysis yields a characterization of the information constrained Pareto frontier in the space of equivalent permanent consumptions. Substituting the expression of $\gamma^P$ from (44) into the equivalent permanent consumptions of agents, in equation (36), and the principal, from equation (39), we obtain

$$\exp(\rho\omega) = \frac{k(\omega)}{y}\exp[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}], \tag{45}$$

where $g$ is the growth rate given in (7), and

$$\exp\rho V(K, \mathbb{P}) = (\rho - \frac{1}{y + \rho\sigma^2 y^3})Kexp[\frac{g}{\rho}]. \tag{46}$$

Equation (45) reflects that each agent consumes a fraction $\frac{1}{y}$ of its capital under management, which grows at average rate $g$, with volatility $\sigma y$ generating a risk premium, and is discounted at rate $\rho$. Similarly (46) reflects that the principal consumes a fraction $(\rho - \frac{1}{y+\rho\sigma^2 y^3})$ of the capital stock, griwng at rate $g$, but is not impacted by any risk, so that unlike in (45), there is no risk premium. As mentioned above, when $\sigma = 0$, there is no incentive problem. Correspondingly (45) and (46) boil down to the first best Pareto frontier in (22).

## 4 Welfare theorems and implementation

Greenwald and Stiglitz [29] have shown that the classical welfare theorems are not valid in an economy with incomplete markets and asymmetric information like ours. In particular, competitive equilibrium allocations are generically constrained inefficient. This is the case in our model. However, our results imply that simple policy interventions can restore the equivalence between information constrained optimal allocations and equilibrium allocations. To facilitate the comparison between our set-up and the welfare theorems, this section focuses on the case where the principal does not consume, i.e., $\gamma^P = 0$.

The implementation of the information-constrained optimum relies on the following institutional arrangements: The principal issues fiat money that can be exchanged against the good at any date $t$ for a price $p_t$, determined in equilibrium.[17] The principal announces a constant tax rate $\tau$ on wealth and sets the growth rate of money supply such that its budget constraint is satisfied.

Our analysis proceeds in three steps. First, we characterize the optimal behavior of agents associated with a given tax rate $\tau$. Second, we determine the associated rational expectations equilibrium and show that, generically, it is not constrained optimal, in accord with Greenwald and Stiglitz [29]. Then we show that optimality can be restored if the principal taxes wealth at an appropriate rate $\tau^*$. In this case the two welfare theorems are valid: any equilibrium is constrained optimal, and conversely, any constrained optimal allocation can be obtained as a competitive equilibrium after lump-sum transfers of initial endowments.

### 4.1 Agents' optimal policy

At $t = 0$, the principal endows an agent with $m_0$ units of money ($M_0$ on aggregate) and commits to a constant tax rate $\tau$ on wealth. We focus on stationary rational expectations equilibria such that

[17] More generally, the principal issues a safe, liquid, short-term asset, which can be traded in response to the $Z^i_t$ shocks. See [24] for further discussion.

$p_t = p_0 exp\pi t$, where $\pi$ is a constant inflation rate. Agents hold capital ($k_t$) and money ($m_t$), so an agent's real wealth at time $t$ is

$$e_t = k_t + \frac{m_t}{p_t}. \tag{47}$$

The dynamics of this real wealth is given by:

$$de_t = k_t(\mu dt + \sigma dZ_t) - [\pi(e_t - k_t) + c_t + \tau e_t]dt. \tag{48}$$

Equation (48) shows that the change in the real wealth of an agent is equal to output minus the sum of the inflation tax, consumption and wealth tax. Since there are no transaction costs, agents can costlessly and continuously rebalance their portfolio of bonds and capital. The only state variable is the agent's aggregate wealth, while $k_t$ and $c_t$ are control variables. Equation (48) and Itô's Lemma imply that the value function $u(e)$ of the agents satisfies the following Bellman equation

$$\rho u(e) = \max_{k,c}[\log c + u'(e)[\mu k - c - \tau e - \pi(e-k)] + \frac{\sigma^2 k^2}{2} u''(e)]. \tag{49}$$

The first order condition with respect to $c$ is

$$\frac{1}{c} = u'(e).$$

The first order condition with respect to $k$ is

$$k = \frac{\mu + \pi}{-\frac{u''(e)}{u'(e)}\sigma^2}.$$

Now suppose that $k_t$ and $c_t$ are a feasible trajectory for a given initial wealth $e$. For any positive constant $\phi$, $\phi k_t$ and $\phi c_t$ are feasible trajectories when $e$ is itself multiplied by $\phi$. This homogeneity property implies that $u(\phi e) = \frac{\log \phi}{\rho} + u(e)$. Taking $\phi e = 1$, we see that the value function is an affine transformation of $\log(e)$:

$$u(e) = \frac{\log(e)}{\rho} + u(1), \tag{50}$$

which implies

$$u'(e) = \frac{1}{\rho e}, u^{''}(e) = -\frac{1}{\rho e^2}. \tag{51}$$

So the first order conditions yield

$$c = \rho e, \tag{52}$$

and

$$k = \frac{\mu + \pi}{\sigma^2} e. \tag{53}$$

That consumption and capital are constant fractions of wealth stems from the logarithmic utility specification. Denoting

$$x := \frac{\mu + \pi}{\sigma^2}, \tag{54}$$

Agents' optimal portfolio choice is to invest a constant fraction $x$ of their wealth in the risky asset and the remaining fraction $1 - x$ in money[18]. As inflation increases, holding money becomes less attractive relative to holding physical capital. Thus, the fraction of wealth invested in the risky asset $x$ is increasing in the inflation rate $\pi$.

[18] In the implementation of the optimal contract in Biais-Mariotti-Plantin-Rochet [8], the continuation pay-off of the agent was "represented" by the cash reserves of the firm managed by the agent: when these cash reserves fell to

## 4.2 Market clearing

In nominal terms, individual demand for money is $m_t = (1-x)p_t e_t$. Since $k_t = xe_t$, individual demand for money is also

$$m_t = \frac{1-x}{x} p_t k_t,$$

so that aggregate demand for money is

$$\frac{1-x}{x} p_t K_t. \tag{55}$$

Market clearing requires that the supply of money by the principal $M_t$ equals aggregate demand. This pins down the equilibrium price of the good $p_t$ which has to be such that:

$$M_t = \frac{(1-x)}{x} p_t K_t. \tag{56}$$

Equation (56) implies that the money supply $M_t$ must grow at the nominal growth rate $\pi + g$, the sum of inflation and real growth rates. Now (52) and (53) imply that

$$g = \mu - \frac{\rho}{x}. \tag{57}$$

and

$$\pi = -\mu + \sigma^2 x. \tag{58}$$

Hence

$$\dot{M}_t = (\sigma^2 x - \frac{\rho}{x}) M_t. \tag{59}$$

Recall that, for simplicity, in this section principal's consumption is equal to 0. So, the only flows of income or expenses for the principal are i) tax revenues or subsidies cost, and ii) seigneurage revenue from the issuance of money or cost of purchasing money back from the agents. Correspondingly, the budget constraint of the principal is:

$$\dot{M}_t = -\tau(p_t K_t + M_t), \tag{60}$$

Now, equation (55) implies that $p_t K_t + M_t = \frac{M_t}{1-x}$. Substituting this equality in (60) and equating to (59), we obtain that

$$\tau = (1-x)(\frac{\rho}{x} - \sigma^2 x). \tag{61}$$

As $x$ goes from 0 to 1, the right-hand side of (61) decreases from infinity to 0. So, for any $\tau > 0$ this equation (61) has a unique solution $x \in (0,1)$, which determines the characteristics of the equilibrium:

**Proposition 8.** *For any tax rate $\tau > 0$, there is a unique rational expectations equilibrium with $\gamma^P = 0$. It is such that:*

- *Agents consume a constant fraction of their wealth, equal to their discount rate $\rho$.*

---

zero, the firm was liquidated. Here the continuation pay-off is "represented" by the wealth $e_t$ of the agent, which is proportional to $exp(\rho\omega_t)$. This is consistent with the property that $\omega_t = u(e_t)$ and the form of the Bellman function: $u(e_t) = \frac{\log e_t}{\rho} + u(1)$. The capital under the agent's management never goes to zero, but poor performance is followed by a reduction in capital.

- *Agents invest a constant fraction $x$ of their wealth in the risky asset, where $x$ is the unique solution of*

$$\tau = (1 - x)(\frac{\rho}{x} - \sigma^2 x). \tag{62}$$

- *Aggregate capital grows at rate $g = \mu - \frac{\rho}{x}$.*

- *The inflation rate is $\pi = -\mu + \sigma^2 x$.*

### 4.3 Extending the two welfare theorems

Proposition 8 states that for any tax rate $\tau > 0$, there is a unique stationary equilibrium allocation. We now analyze the link between these equilibrium allocations and information-constrained optimal allocations.

To do so, the following table compares capital allocation, consumption, growth rate, and capital dynamics in equilibrium and in constrained optima with $\gamma^P = 0$:

| | Constrained Optima | Equilibria |
|---|---|---|
| capital allocations | $k(\omega_t) = \frac{\exp(\rho\omega_t)}{a}$ | $k_t = xe_t$ |
| consumption | $c_t = \frac{k_t}{y}$ | $c_t = \rho e_t$ |
| growth rate | $g(y) = \mu - \frac{1}{y}$ | $g = \mu - \frac{\rho}{x}$ |
| capital dynamics | $\frac{dk_t}{k_t} = g(y)dt + \rho\sigma y dZ_t$ | $\frac{dk_t}{k_t} = gdt + \sigma x dZ_t$ |

The first two lines of the table emphasize that an agent's capital and consumption are proportional to this agent's equivalent permanent consumption in the optimal mechanism, while they are proportional to the agent's wealth in the equilibrium. So, allocations will be the same in the optimal mechanism and in equilibrium if the dynamics of equivalent permanent consumption and that of wealth are the same.

As can be seen in the table, growth rates and capital dynamics are the same in the equilibrium with risk exposure $x$ and in the information-constrained optimum parametrized by $y$ if and only if $x = \rho y$. Once the dynamics of capital is the same in the equilibrium as in the information-constrained optimum, for a given distribution of initial capital, capital remains the same at all time in the equilibrium as in the information-constrained allocation. Once capital is the same in the equilibrium and in the information-constrained optimum, the expressions for capital and consumption in the table imply that consumption is the same in the equilibrium and in the information constrained optimum. We thus obtain our next proposition (whose proof is in the appendix):

**Proposition 9.** *Denote by $x_{min}$ the only positive solution of the cubic equation $\frac{\sigma^2}{\rho}x^3 + x = 1$, and let*

$$\tau^* = (1 - x_{min})(\frac{\rho}{x_{min}} - \sigma^2 x_{min}).$$

*When the principal does not consume:*

1. *If the principal taxes wealth at rate $\tau^*$ and has zero consumption, the competitive equilibrium is constrained Pareto optimal for any initial distribution of capital and money.*

2. *Conversely, any information-constrained Pareto optimal allocation with $\gamma^P = 0$ can be implemented as an equilibrium allocation, after initial lump-sum transfers, and with tax rate $\tau^*$.*

The first point in the proposition is the counterpart, in our setting, of the first theorem of welfare, while the second point is the counterpart of the second theorem of welfare.

The proposition clarifies that in our economy the role of taxes is neither to finance government expenditures (we set principal consumption to 0), nor to redistribute initial wealth between rich and poor agents (which is done through lump-sum transfers at $t = 0$), but to enable risk-sharing among agents exposed to different productivity shocks $dZ_t^i$. In line with Gordon [28], wealth taxes in our economy can be interpreted as a to way finance social insurance, understood as risk-sharing among agents exposed to different risks.

While Proposition 8 states that for any tax rate $\tau$ there exists a competitive equilibrium, Proposition 9 states that there is a unique tax rate $\tau^*$ (and correspondingly a unique portfolio structure $x_{min}$) for which the equilibrium allocation is constrained Pareto optimal. This implies that, in our framework, laissez faire is generically not constrained Pareto optimal. To see this more precisely, first note that laissez faire corresponds to $\tau = 0$. Now, by Proposition 8, there are two possible equilibrium values of $x$ corresponding to zero taxation: $x = 1$ and $x = \frac{\sqrt{\rho}}{\sigma}$. For these values of $x$ to correspond to an information-constrained optimum they must be equal to $x_{min}$. $x_{min} = 1$ if and only if $\sigma = 0$. That is, no taxation is Pareto optimal if agents are not exposed to any idiosyncratic risk, which is consistent with our above remark that taxation is helpful to implement risk-sharing. But $\sigma = 0$ is not generic. Similarly, $x_{min} = \frac{\sqrt{\rho}}{\sigma}$ if and only if $\frac{\sqrt{\rho}}{\sigma} = .5$, which also is not generic.

# 5 Conclusion

This paper characterizes optimal capital allocation and risk sharing between a principal and many agents, who privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [10]. To provide agents with incentives for truthful revelation, the optimal dynamic mechanism increases agents' continuation utility after good reported performance and reduces it after bad reported performance. In this context, agents' continuation utilities are random variables and their distribution across agents is a state variable of the problem faced by the principal. This gives rise to a Bellman equation in an infinite dimensional space. To solve this difficult problem we rely on mean-field techniques. Under the assumption of logarithmic we fully characterize the optimal dynamics of capital and consumption as well as the distribution of continuation utilities across agents.

Moreover, we show that the optimal dynamic mechanism can be implemented in equilibrium with a market in which agents trade goods for money, issued by the principal, and wealth taxation An appropriately set money supply growth rate gives rise to an optimal inflation rate, inducing agents to allocate their wealth among capital and money in a way that gives rise to the same risk exposure as in the optimal mechanism. Wealth taxation complements money to engineer in equilibrium the same level of risk-sharing as in the optimal mechanism.

This implementation result is akin to the second welfare theorem: For any Pareto optimal allocation, there exists a combination of fiscal and monetary policies implementing that allocation in equilibrium. However, while in the classical welfare theorem, markets are perfect and complete, in our analysis markets are endogenously incomplete because of information asymmetry. Moreover, while in the classical second welfare theorem, any proportional tax is distortive, in our analysis, proportional wealth taxes optimally affect agents' behaviour. Finally note that the first theorem of welfare also has to be modified in our context. Only a subset of the equilibria arising in our setting are information-constrained Pareto optima. In particular, the laissez-faire equilibrium, obtained with no taxation and no public expenditure, is not constrained Pareto optimal.

# Appendix A: Proofs

**Proof of Lemma 1** The problem can be solved by recursive methods, which we develop in Section 3. Here, we provide a heuristic derivation of the solution in terms of Lagrange multipliers. Denoting by $\lambda(\cdot)$ and $\eta$ the Lagrange multipliers associated with the infinite-dimensional constraints (11) and (12), respectively, the *Lagrangian* is

$$L = \int_0^\infty e^{-\rho t}\left[\log c^P(t,K_t,\mathbb{P}) - \int_{\mathbb{R}} \lambda(\omega)\log \bar{c}^A(t,K_t,\omega,\mathbb{P})\,d\mathbb{P}(\omega) - \eta e^{-(\mu-\rho)t}(c^P(t,K_t,\mathbb{P}) + c^A(t,K_t,\mathbb{P}))\right] dt \\ + \eta K.$$

By assuming sufficient integrability of the functions $c^P(t,K_t,\mathbb{P})$ and $\bar{c}^A(t,K_t,\omega,\mathbb{P})$, the first order conditions of this problem are for every $t \geq 0$ and every $\omega \in \mathbb{R}$,

$$c^P(t,K_t,\mathbb{P}) = \frac{1}{\eta}e^{(\mu-\rho)t} \quad \text{and} \quad \bar{c}^A(t,K_t,\omega,\mathbb{P}) = -\frac{\lambda(\omega)}{\eta}e^{(\mu-\rho)t}.$$

Since $K_0 = K$, we have for every $t \geq 0$ and every $\omega \in \mathbb{R}$,

$$c^P(t,K_t,\mathbb{P}) = e^{(\mu-\rho)t}c^P(0,K,\mathbb{P}), \tag{63}$$
$$\bar{c}^A(t,K_t,\omega,\mathbb{P}) = e^{(\mu-\rho)t}\bar{c}^A(0,K,\omega,\mathbb{P}). \tag{64}$$

Integrating (64) by means of (8) yields

$$c^A(t,K_t,\mathbb{P}) = e^{(\mu-\rho)t}c^A(0,K,\mathbb{P}) \tag{65}$$

Inserting (63) and (65) into (12) and using (63) and (65) again, we get

$$c^P(t,K_t,\mathbb{P}) + c^A(t,K_t,\mathbb{P}) = \rho K e^{(\mu-\rho)t}. \tag{66}$$

Inserting (66) into the ODE (9) now yields

$$\dot{K}_t = \mu K_t - \rho K e^{(\mu-\rho)t},$$

which has the solution

$$K_t = K e^{(\mu-\rho)t}. \tag{67}$$

(66) then can be written in the recursive formulation

$$c^P(t,K_t,\mathbb{P}) + c^A(t,K_t,\mathbb{P}) = \rho K_t, \tag{68}$$

which means that aggregate consumption is a constant fraction $\rho$ of aggregate capital $K$. This property reflects our assumptions that the agents and the principal have logarithmic utility.

Next, combining (65) and (11), integrating partially, and re-arranging yields the consumption of each agent

$$\bar{c}^A(t,K_t,\omega,\mathbb{P}) = e^{(\mu-\rho)t}e^{-\frac{\mu-\rho}{\rho}}e^{\rho\omega} \tag{69}$$

Using (67), (69), and (68), the lemma obtains. QED

**Proof of Proposition 1** Inserting $\bar{c}^A$ from (69) into (18) and verifying yields

$$\omega_t = \omega + \frac{\mu - \rho}{\rho} t.$$

QED

**Proof of Lemma 2** We omit the index $i$ to alleviate notations. We will apply the martingale optimality principle to the process family indexed by $\Delta$ defined as

$$R_t^\Delta = e^{-\rho t}\omega_t + \int_0^t e^{-\rho s} \log(c_s + \sigma k_s \Delta_s)\, ds,$$

where

$$d\omega_t = (\rho\omega_t - \log(c_t))\, dt + \sigma y_t\, d\hat{Z}_t.$$

We have

$$\begin{aligned} dR_t^\Delta &= e^{-\rho t}(y_t d\hat{Z}_t + (\log(c_t + \sigma k_t \Delta_t) - \log(c_t))\, dt \\ &= e^{-\rho t} y_t dZ_t + (\log(c_t + \sigma k_t \Delta_t) - \log(c_t) - y_t \Delta_t)\, dt \end{aligned}$$

Because log is concave, we observe

$$\log(c_t + \sigma k_t \Delta_t) - \log(c_t) - y_t \Delta_t \leq \Delta \left( \frac{\sigma k_t}{c_t} - y_t \right) \leq 0.$$

Therefore, the process $(R_t^\Delta)_t$ is a $\mathbb{P}$ supermartingale for every nonnegative process $\Delta 0$ and a martingale for the thruthful report case $\Delta = 0$. Thus, we have

$$R_0 = \omega_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho s} \log(c_s) ds\right] \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s} \log(c_s + \sigma k_s \Delta_s) ds\right],$$

which implies the optimality of a truthful report. QED

**Proof of Proposition 2:** Let $\alpha = (c, c^P, y)$ be an admissible feedback control. We denote

$$J_\lambda^\alpha = \int_0^\infty e^{-\rho t} \left( \log c_t^P + \lambda(K_t, \mathbb{P}_t) \left( K_t - \int y_t(\omega) c_t(w) d\mathbb{P}_t(\omega) \right) \right) dt,$$

and

$$J^\alpha = \int_0^\infty e^{-\rho t} \log(c_t^P)\, dt.$$

For every Lagrange multiplier $\lambda$, we have by assumption i),

$$V_\lambda = J_\lambda^{\alpha_\lambda} \geq J_\lambda^\alpha.$$

In particular, for $\lambda = \lambda_0$,

$$V_{\lambda_0} = J_{\lambda_0}^{\alpha_{\lambda_0}} = J^{\alpha_{\lambda_0}}$$

On the other hand, for $\alpha \in \mathcal{K}$, we have $V_{\lambda_0} = J_{\lambda_0}^{\alpha_{\lambda_0}} \geq J_{\lambda_0}^\alpha = J^\alpha$ yielding $V_{\lambda_0} \geq \sup_{\alpha \in \mathcal{K}} J^\alpha$. Because $\alpha_{\lambda_0} \in \mathcal{K}$, $V = V_{\lambda_0}$ and the proof is complete.

QED

**Proof of Proposition 3:** If the value $V_\lambda$ is regular, the results follows from a direct applicatio of the Itô's formula (81) given in Appendix B.

**Proof of Proposition 4:** Fix $\mathbb{P} \in \mathbb{P}_2(\mathbb{R})$ and a Lagrange multiplier $\lambda$. Let $\mathbb{P}_t$ be the probability distribution of the process $\omega_t$ when the initial probability distribution of $\omega_0$ is $\mathbb{P}$. Let us consider some arbitrary feedback control $\alpha(K_t, \mathbb{P}_t, \omega_t)$. We apply Itô's formula (81) to $v^\lambda(K_t, \mathbb{P}_t)$ between $s = 0$ and $s = t$ for $t > 0$.

$$\begin{aligned}
e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) &= v^\lambda(K, \mathbb{P}) \\
&\quad + \int_0^t e^{-\rho s} \left( -\rho v^\lambda(K_s, \mathbb{P}_s) + v_K^\lambda(K_s, \mathbb{P}_s) \left( \mu K_s - c^P(K_s, \mathbb{P}_s) - \int c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds \\
&\quad + \int_0^t e^{-\rho s} \int \partial_\omega \delta v^\lambda[K_s, \mathbb{P}_s](\omega)(\rho\omega - \log c(K_s, \mathbb{P}_s, \omega)) d\mathbb{P}_s(\omega)\, ds \\
&\quad + \int_0^t e^{-\rho s} \int \partial_{\omega\omega} \delta v^\lambda[(K_s, \mathbb{P}_s](\omega) \frac{\sigma^2}{2} y^2(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega)\, ds.
\end{aligned}$$

We deduce from the Bellman equation (34) satisfied by $v^\lambda$ that

$$\begin{aligned}
v^\lambda(K, \mathbb{P}) &\geq e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) \\
&\quad + \int_0^t e^{-\rho s} \left( \log(c^P(K_s, \mathbb{P}_s) + \lambda(K_s, \mathbb{P}_s) \left( K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.
\end{aligned}$$

Letting $t$ tend to $+\infty$ and using the transversality condition, we obtain

$$v^\lambda(K, \mathbb{P}) \geq \int_0^\infty e^{-\rho s} \left( \log(c^P(K_s, \mathbb{P}_s)) + \lambda(K_s, \mathbb{P}_s) \left( K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds = J_\lambda^\alpha.$$

Since the control $\alpha$ is arbitrary, we obtain $v^\lambda(K, \mathbb{P}) \geq V_\lambda$. On the other hand, let us apply the same Itô argument with the control $\alpha_\lambda^*$ attaining the maximum in (34). We obtain

$$v^\lambda(K, \mathbb{P}) = J_\lambda^{\alpha_\lambda^*} \leq V_\lambda,$$

which yields that $v^\lambda = V_\lambda$. We conclude the proof by applying Proposition 2.
QED

**Proof of Proposition 5:** Denote $M_t \equiv \int exp(\rho\omega) d\mathbb{P}$. To obtain the dynamics of $M_t$, we substitute $\gamma^A = 1/(ym)$ in $c(\omega) = \gamma^A \exp(\rho\omega)$, and then substitute the resulting expression into (31), which yields

$$d\omega_t = \log(ym)dt + \sigma y\, dZ_t. \tag{70}$$

(70) and $M_t = \mathbb{E}[\exp(\rho\omega_t)]$ yield

$$M_t = M_0 \mathbb{E}\left[\exp(\rho\left(\log(ym)t + \sigma y Z_t\right))\right] = M_0 \exp\left(\left(\rho \log(ym) + \frac{\rho^2\sigma^2 y^2}{2}\right) t\right), \tag{71}$$

which gives

$$\frac{dM_t}{M_t} = \left(\rho \log(ym) + \frac{\rho^2\sigma^2 y^2}{2}\right) dt. \tag{72}$$

Equality of the growth rates of $K_t$ and $M_t$ means that

$$\mu - \gamma^P - \frac{1}{y} = \rho \log(ym) + \frac{\rho^2\sigma^2 y^2}{2}. \tag{73}$$

The restricted principal's problem is thus characterized by the following maximization problem:

$$V(K, \mathbb{P}) = \sup_{\gamma_P, y} \int_0^{+\infty} e^{-\rho t} \log(\gamma^P K_t) dt, \tag{74}$$

under the constraint (73) and the dynamics of capital

$$K_t = K \exp((\mu - \gamma^P - \frac{1}{y})t). \tag{75}$$

Substituting $K_t$ from (75) into (74), the latter writes

$$V(K, \mathbb{P}) = \sup_{\gamma_P, y} \int_0^{+\infty} [e^{-\rho t} \left( \log(\gamma_P K) + (\mu - \gamma^P - \frac{1}{y})t \right)] dt, \text{ s.t., (73)}. \tag{76}$$

Easy computations then show that (76) can be rewritten as

$$\rho V(K, \mathbb{P}) = \log K + \sup_{\gamma_P, y} \left( \log \gamma_P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} \right), \text{ s.t., (73)}. \tag{77}$$

Using (73) we can express $\gamma^P$ as a function of $y$ and $m$

$$\gamma^P = \mu - \frac{1}{y} - \rho \left( \log(ym) + \frac{\rho\sigma^2 y^2}{2} \right) = \psi(y) - \rho \log(m).$$

Substituting the value of $\gamma^P$ into (77), the latter writes as

$$\rho V(K, \mathbb{P}) = \log K + \sup_{y} \left( \log \left( \mu - \frac{1}{y} - \rho \left( \log(yu) + \frac{\rho\sigma^2 y^2}{2} \right) \right) + \log(ym) + \frac{\rho\sigma^2 y^2}{2} \right). \tag{78}$$

There exists a solution to (78) when the feasible set is non empty, i.e. when it is possible to find values of $y$ for which the argument of the first log is positive. This is equivalent to $m \leq m_{max}$. Taking the first order condition in (78) and denoting

$$\rho v^\star(m) := \sup_{y} \left( \log \left( \mu - \frac{1}{y} - \rho \left( \log(ym) + \frac{\rho\sigma^2 y^2}{2} \right) \right) + \log(ym) + \frac{\rho\sigma^2 y^2}{2} \right), \tag{79}$$

we obtain that $\rho V(K, \mathbb{P}) = \log K + \rho v^\star(m)$. QED

**Proof of Proposition 6:** The proof consists in two steps. First, we have to prove the existence of a Lagrangian multiplier $\lambda_0$, such that the feasibility constraint (33) is satisfied. Second, we have to prove that $v^*$ satisfies the transversality condition for the class of admissible controls defined below to apply Theorem 4.

*Step 1:* The first order conditions in (42) give

$$-\lambda K y - \left( \frac{1}{\rho} - m v'(m) \right) - \rho \frac{v'(m)}{\gamma^A} = 0$$

and

$$-\lambda K\gamma_A + \rho^2\sigma^2 y v'(m) = 0.$$

Solving the two equations above gives the optimal controls $y^*(\lambda)$ and $\gamma^A(\lambda)$,

$$y^*(\lambda) = \frac{-\rho\sigma^2\left(\frac{1}{\rho} - mv'(m)\right) + \sqrt{\rho^2\sigma^2\left(\frac{1}{\rho} - mv'(m)\right)^2 - 4\rho\sigma^2\lambda^2K^2}}{2\lambda K\rho\sigma^2}$$

and

$$\gamma^{*,A}(\lambda) = \frac{\rho^2\sigma^2 v'(m)}{\lambda K} y^*(\lambda).$$

A tedious computation shows that the feasibility constraint $\gamma^{*,A}(\lambda)y^*(\lambda)m = 1$ gives a cubic equation for $\lambda$ that admits a solution.

*Step 2:* For $\varepsilon > 0$, let us define the set

$$\mathcal{A}_\varepsilon = \{\alpha \text{ admissible feedback controls s.t.} \int \exp(\rho\omega)d\mathbb{P}_t^{(\alpha)}(\omega) \leq (m_{max} - \varepsilon)K_t^{(\alpha)} \text{ for all } t \geq 0\}$$

and define the set $\mathcal{A}$ as the union of $\mathcal{A}_\varepsilon$. We will prove that for every control $\alpha \in \mathcal{A}$, we have the transversality condition $\lim_{t\to+\infty} e^{-\rho t}\left(\frac{\log K_t^\alpha}{\rho} + v^\star(M_t^\alpha)\right) = 0$ where

$$M_t^\alpha = \frac{\int \exp(\rho\omega)d\mathbb{P}_t^{(\alpha)}(\omega)}{K_t^{(\alpha)}}.$$

Take $\alpha \in \mathcal{A}$. There is $\varepsilon > 0$, such that $\alpha \in \mathcal{A}_\varepsilon$. Because $v^\star$ is continuous, $v^\star$ is bounded by a constant $C_\varepsilon$ on the interval $[0, m_{max} - \varepsilon]$, and we have

$$e^{-\rho t}\left(\frac{\log K_t^\alpha}{\rho} + v^\star(M_t^\alpha)\right) \leq e^{-\rho t}\frac{\mu}{\rho}t + e^{-\rho t}C_\varepsilon,$$

which proves the transversality condition. QED

**Proof of Proposition 7:** To prove Point 1 in Proposition 7 we start by observing that (30) states that the growth rate of capital is

$$g = \mu - \frac{\int c(\omega)d\mathbb{P}(\omega)}{K} - \frac{c^P}{K}$$

and that (3.4.1) states that

$$c^P = \gamma^P K, c(\omega) = \gamma^A \exp(\rho\omega).$$

Substituting the latter in the former, we have

$$g = \mu - \frac{\gamma^A \int \exp(\rho\omega)d\mathbb{P}(\omega)}{K} - \gamma^P.$$

By the definition of $M_t$, this is

$$g = \mu - \gamma^A\frac{M}{K} - \gamma^P. \tag{80}$$

As explained in the analysis of the restricted problem, (3.4.1) implies $\frac{M_t}{K_t}$ is a constant, denoted by $m$, and $\gamma^A = \frac{1}{ym}$. Substituting in (80) yields

$$g = \mu - \frac{1}{y} - \gamma^P.$$

Substituting $\gamma^P$, we obtain Point 1 in Proposition 7.

To prove Point 2 in Proposition 7, we start by recalling that (70) states

$$d\omega = \log(ym)dt + \sigma y dZ_t$$

and that (73) implies

$$\log(ym) = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Noting that the first term on the right-hand side is $\frac{g}{\rho}$, we obtain Point 2 in Proposition 7.

Point 3 in Proposition 7 is just a restatement of $\gamma_P$, while Points 4 and 5 are established at the beginning of the analysis of the restricted problem.

QED

**Proof of Proposition 9** Proposition 7 implies that the value of $y$ corresponding to an information-constrained Pareto optimum in which $\gamma^P = 0$ is the root of

$$\rho\sigma^2 y^3 + y = \frac{1}{\rho}.$$

Substituting $x = \rho y$ in this equation, we obtain

$$\frac{\sigma^2}{\rho}x^3 + x = 1,$$

whose unique positive root is $x_{min}$, as stated in Proposition 9. By Proposition 8 this is the portfolio structure chosen by agents in equilibrium when the tax rate is that stated in Proposition 9

$$\tau^* = (1 - x_{min})(\frac{\rho}{x_{min}} - \sigma^2 x_{min}).$$

This establishes the second part of proposition 9. To establish the first part, distribute initial capital proportionally to $exp(\rho\omega)$ and set $\tau = \tau^*$.

QED

# Appendix B: Differential calculus in the Wasserstein space

Consider a real-valued function $F$ defined on $\mathcal{P}_2(\mathbb{R})$, which is the set of probability measures on $\mathbb{R}$ with finite second moment. We endow $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein distance. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $\Pi(\mu, \nu)$ is the set of transport plans, that is, probability measures on $\mathbb{R} \times \mathbb{R}$ with respective marginals $\mu$ and $\nu$. The Wasserstein distance $W_2$ on $\mathcal{P}_2(\mathbb{R})$ is defined as the square root of

$$\min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} |y - x|^2 \, d\gamma(x, y).$$

To apply a verification argument for the principal problem, we are interested in Itô's formula for $F$ to describe the dynamic $t \to F(\mathbb{P}_t)$, where $\mathbb{P}_t$ is the marginal probability distribution of the process $(w_t)_t$ given by Equation (31). Itô's formula for $F$ naturally requires differential calculus on the space of measures. We start by introducing the two notions of first variation and L-differentiability for functions of measures relying on the convexity of $\mathcal{P}_2(\mathbb{R})$ (for a more rigorous treatment, see for instance the standard references ([13] Definition 5.43 and Proposition 5.48), [12] and [49]).

**Definition 1.** *We will say that*

- *A function $F$ admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued and continuous function $\delta F[\mu] : \mathbb{R} \to \mathbb{R}$, such that for all $\nu$ in $\mathcal{P}_2(\mathbb{R})$, we have*
$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( F((1-\varepsilon)\mu + \varepsilon\nu) - F(\mu) \right) = \int_{\mathbb{R}} \delta F[\mu](\omega) \, d(\nu - \mu)(\omega).$$
- *A function $F$ is L-differentiable at $\mu \in \mathcal{P}_2(\mathbb{R})$ if the function $\delta F[\mu]$ is twice differentiable on $\mathbb{R}$ and we will denote by $\partial_x \delta F[\mathbb{P}]$ and $\partial_{xx} \delta F[\mathbb{P}]$ its first and second L-derivatives.*

For a function $F$ that is L-differentiable, Itô's formula, associated to the dynamic $t \to F(\mathbb{P}_t)$ with the dynamics of $\omega_t$ given in (31), takes the following form (see [13], Chapter 5, Th. 5.99):

$$\begin{aligned} F(\mathbb{P}_t) \;\; &= F(\mathbb{P}_0) + \textstyle\int_0^t \mathbb{E}\left[ \partial_x \delta F[\mathbb{P}_s](w_s)(\rho w_s - \log c^A(K_s, \mathbb{P}_s, w_s)) \right] \, ds \\ &\quad + \tfrac{1}{2} \textstyle\int_0^t \mathbb{E}\left[ \partial_{xx} \delta F[\mathbb{P}_s](w_s) \sigma^2 y^2(K_s, \mathbb{P}_s, w_s) \right] \, ds. \end{aligned} \tag{81}$$

A general class of L-differentiable functions can be described as follows. Let $\phi$ be a twice continuously differentiable function on $\mathbb{R}$ with quadratic growth and $v$ a continuously differentiable function on $\mathbb{R}$. We consider the function $F$ defined on $\mathcal{P}_2(\mathbb{R})$ by

$$F(\mu) = v\left( \int_{\mathbb{R}} \phi(x)\mu(dx) \right).$$

Then, $F$ is L-differentiable with

$$\delta F[\mu] = v'\left( \int_{\mathbb{R}} \phi(x)\mu(dx) \right) \phi, \; \partial_x \delta F[\mu] = v'\left( \int_{\mathbb{R}} \phi(x)\mu(dx) \right) \phi' \text{ and } \partial_{xx} \delta F[\mu] = v'\left( \int_{\mathbb{R}} \phi(x)\mu(dx) \right) \phi''.$$

# Appendix C: Existence of the mean-field limit

The objective of this technical note is to clarify the existence of the limit equations (30) and (31) by adapting for sake of completeness the techniques developed in [48].
To simplify the exposition, we consider that the function $y$ is constant and we normalize it to 1. The proof below can be easily extended to the case where the function $y$ is Lipschitz by applying the Burkholder, Davis and Gundy inequality.
Equations (30) and (31) can be summarized by the McKean Vlasov SDE

$$\begin{cases} d\omega_t = & b_1(\omega_t, K_t, \mu_t) \, dt + dZ_t \\ dK_t = & b_2(K_t, \mu_t) \, dt, \end{cases} \tag{82}$$

where $\mu_t$ is the probability distribution of $\omega_t$ for all $t$ and the two functions $b_i$ are Lipschitz on $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R})$ respectively.

**Proposition 10.** *There exists a unique strong solution of the McKean Vlasov equation (82).*

**Proof** Fix $T \geq 0$. Define the truncated supremum norm $||x|| = \sup_{s\in[0,t]} |x_s|$ for $x \in \hat{C} = C([0,T],\mathbb{R})$. We define the following truncated distance on $\mathcal{P}_2(\hat{C})$,

$$d_t^2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\hat{C}\times\hat{C}} ||x-y||_t \pi(dx,dy).$$

Fix $\mu \in \mathcal{P}_2(\hat{C})$ and consider the SDE

$$\begin{cases} d\omega_t^\mu = & b_1(\omega_t^\mu, K_t^\mu, \mu_t)\,dt + dZ_t \\ dK_t^\mu = & b_2(K_t^\mu, \mu_t)\,dt, \end{cases} \tag{83}$$

Because the function $b_i$ are Lipschitz, there exists a unique square-integrable solution $(\omega^\mu, K^\mu)$ to (83). Define the map $\Phi : \mathcal{P}_2(\hat{C}) \times \mathcal{P}_2(\hat{C})$ that assigns to $\mu$ the probability distribution of $\omega^\mu$. Observe that fixed points of $\Phi$ are solutions of the McKean Vlasov equation (82).
Let $\mu,\nu \in \mathcal{P}_2(\hat{C})$, for $s \leq t \leq T$, we have using the Cauchy-Schwarz inequality

$$|K_s^\mu - K_s^\nu| \leq s \int_0^s |b_2(K_u^\mu, \mu_u) - b_2(K_u^\nu, \nu_u)|^2\,du.$$

Using the Lipschitz assumption, the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and taking the supremum over $[0,t]$, we obtain

$$\mathbb{E}\left[||K^\mu - K^\nu||_t^2\right] \leq 2tL^2 \mathbb{E}\left(\int_0^t \left(||K^\mu - K^\nu||_s^2 + W_2^2(\mu_s,\nu_s)\right)\,ds\right).$$

Using Fubini's theorem and Gronwall's inequality, we obtain

$$\mathbb{E}\left[||K^\mu - K^\nu||_t^2\right] \leq C\int_0^t W_2^2(\mu_s,\nu_s)\,ds \leq C\int_0^t d_s^2(\mu,\nu)\,ds. \tag{84}$$

Repeating the same argument, we have

$$\begin{aligned} \mathbb{E}\left[||\omega^\mu - \omega^\nu||_t^2\right] &\leq 2tL^2\mathbb{E}\left(\int_0^t \left(||\omega^\mu - \omega^\nu||_s^2 + ||K^\mu - K^\nu||_s^2 + W_2^2(\mu_s,\nu_s)\right)\,ds\right) \\ &\leq 2tL^2\mathbb{E}\left(\int_0^t \left(||\omega^\mu - \omega^\nu||_s^2 + W_2^2(\mu_s,\nu_s)\right)\,ds\right) + 2t^2L^2\mathbb{E}\left[||K^\mu - K^\nu||_t^2\right] \\ &\leq C\mathbb{E}\left(\int_0^t \left(||\omega^\mu - \omega^\nu||_s^2 + W_2^2(\mu_s,\nu_s)\right)\,ds\right), \end{aligned}$$

where the last inequality uses (84). Using Fubini's theorem and Gronwall's inequality again, we obtain

$$\mathbb{E}\left[||\omega^\mu - \omega^\nu||_t^2\right] \leq C\int_0^t d_s^2(\mu,\nu)\,ds.$$

By definition of $d_t^2$ and observing that the joint distribution of $(\omega^\mu, \omega^\nu)$ is a coupling, we finally obtain

$$d_t^2(\Phi(\mu),\Phi(\nu)) \leq \mathbb{E}\left[||\omega^\mu - \omega^\nu||_t^2\right] \leq C\int_0^t d_s^2(\mu,\nu)\,ds.$$

The proof of existence and uniqueness now follows from the usual Picard iteration procedure.

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