Working paper

ACTING IN THE DARKNESS: TOWARDS SOME FOUNDATIONS FOR THE *PRECAUTIONARY PRINCIPLE*

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Abstract. Invoked to guide actions under irreversibility and uncertainty, the *Precautionary Principle* states that decision-makers should act cautiously unless the consequences of acts are known. We consider a setting where the stock of past actions, passed a tipping point which remains unknown, increases the probability of a catastrophe. When past acts are observable, decision-makers can reconstruct the whole evolution of stock and beliefs and follow an optimal trajectory. Otherwise, and in accordance with the Precautionary Principle, they act cautiously, remaining too optimistic on their ability to delay the tipping point. This suboptimal behaviour has minor consequences on welfare.

Keywords. *Precautionary Principle*, Environmental Risk, Tipping Point, Uncertainty and Irreversibility.

JEL codes. D83, Q55.

1. INTRODUCTION

On the Precautionary Principle. The major environmental and health issues that pertain to our modern *risk society* are most often due to our own production and consumption.[1] When dealing with such risks, decision-making is complicated by two features that make the standard tools of cost-benefit analysis of limited value. The first specificity is that consumption and production choices might entail irreversibility. The most salient example is given by global warming. Pollutants have been accumulating in the atmosphere from the beginning of the industrial era, leading to a steady increase in temperature. All current or planned efforts against global warming consist in controlling the growth rate of temperature, with little hope of reducing it. The second feature of those problems is that the costs and benefits of any decision have to be assessed under significant uncertainty. Although the consequences of acting might be detrimental to the

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[1] See Beck (1992).

environment, the extent to which it is so and the probability of harmful events remain to a large extent unknown to decision-makers when acting.

To guide decision-making in such contexts, the so called *Precautionary Principle* has been repeatedly invoked. The original idea is due to philosopher Hans Jonas. His *Principle of Foresight* states that decision-makers should recognize the long-term irreversible consequences of their current actions, and refrain from undertaking any such action if there is no proof that it would not negatively affect future generations' well-being.[2]

Since its inception, there has always been a lively debate, mainly led by philosophers and political scientists, on whether the *Precautionary Principle* offers a convenient guide for decision-making under uncertainty. On the one hand, the fact that it serves as a background for some regulatory policies suggests that it should be judged on normative grounds. On the other hand, that doubts always exist on the fact that its adoption might actually do more harm, by hindering innovation and growth, than good, by protecting human health or the environment points at the more positive view of this notion and suggests that the *Precautionary Principle* might just describe suboptimal behavior.[3]

This paper proposes a simple model of dynamic decision-making under irreversibility and uncertainty that aims at giving theoretical foundations for the *Precautionary Principle* and assesses its relevance in practice. Hereafter, an action (consumption/production) taken at any point in time yields a flow payoff. The stock of past actions affects the arrival rate of an environmental catastrophe. This catastrophe is a major disruptive event with all opportunities for consumption/production disappearing afterwards.[4] Passed a tipping point, this arrival rate irreversibly jumps up.[5] Only the distribution of possible tipping points is known.[6] Whether the tipping point has been passed or not is ignored by the decision-maker (thereafter DM). Acting today changes how likely it is that the tipping point will be passed in the near future and thus affects posterior beliefs in case no catastrophe takes place. In this context, an optimal trajectory should *a priori* follow a feedback rule that stipulates actions in terms of two state variables: the level of stock and the decision-maker's beliefs on whether the tipping point has been passed. As DM becomes more pessimistic and believes that it is more likely that the tipping point has been passed, choosing actions closer to the myopic optimum becomes more attractive.

[2]The *Precautionary Principle* was acknowledged by the United Nations in 1992, during the *Rio Earth Summit*, and perhaps expressed less restrictively as: *"Where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."* A similar principle was invoked in the French 2004 *Charter on Environment* (Loi constitutionnelle n 2005-205 du 1 mars 2005 relative à la Charte de l'environnement) that is now part of the French Constitution. Any risk, health or environmental regulation must thus comply with the legal framework that the *Precautionary Principle* contributes to build.

[3]See Sunstein (2005), Gardiner (2006), Giddens (2011), O'Riordan (2013) for informal discussions and Immordino (2003) for a survey of the relevant literature for economics.

[4]See Cropper (1976), Gjerde et al. (1999) and Clarke and Reed (1994) for a similar assumption.

[5]Tipping points models are frequently used in ecology and in climatology (Lenton et al., 2008). To illustrate, a recent report by the World Bank argues that *"As global warming approaches and exceeds 2-degrees Celsius, there is a risk of triggering nonlinear tipping elements. Examples include the disintegration of the West Antarctic ice sheet leading to more rapid sea-level rise. The melting of the Arctic permafrost ice also induces the release of carbon dioxide, methane and other greenhouse gases which would considerably accelerate global warming."* See http://whrc.org/project/arctic-permafrost.

[6]Kriegler et al. (2009) offers a view of what experts might think of those distributions of tipping points. Roe and Baker (2007) argues that whether past actions have already triggered a change of regimes might remain unknown for a while.

Stock-Markov Equilibria. This optimal path helps to understand how trajectories are modified under more realistic assumptions on how a society addresses such dynamic problems. In this respect, we view an ongoing society as a game where different selves of DM may act at different points in time. In a *Stock-Markov Equilibrium* (thereafter SME), those selves adopt a simple feedback rule only based on the level of the stock. Each self can only commit to an action over an infinitesimal period of time (a so called *impulse deviation*); anticipating that future selves abide to the same *Stock-Markov* feedback rule. Of course, the evolution of beliefs along the equilibrium path is not only consistent with the feedback rule but also with the underlying information structure.

Observable Impulse Deviations. Suppose first that past impulse deviations are observable by future selves. In any such *SME*, future selves will certainly believe that the tipping point is more likely to have been passed following a past deviation that has increased the stock they inherited; a *Pessimistic Stigma.* Thinking that the tipping point is more likely to have been passed, yet no catastrophe has occured, future selves no longer adopt a safe stance and actions jump towards the myopic optimum.

Implementation of the Optimum. An optimal trajectory can always be implemented as a *SME* when impulse deviations are observable. Although an optimal feedback rule defines actions in terms of stock and beliefs, those state variables evolve on a one-dimensional manifold along the optimal trajectory. The optimal feedback rule thus induces a *Stock-Markov* feedback rule on path. By construction, actions being the same with those two rules, beliefs evolve similarly. Off path, future selves always reconstruct the evolution of beliefs from the observed past impulse deviation of a predecessor and the conjecture that, beyond such deviation, all selves abide to the equilibrium feedback rule.

Non-Observable Impulse Deviations. In contrast, consider the more realistic scenario where impulse deviations cannot be detected by future selves. This scenario stands as a metaphor for the case where the consequences of past actions cannot be inferred in the future. An informational externality now arises across decision-makers. Future selves can no longer infer that the tipping point is more likely to have been passed if they have not been able to observe past impulse deviations that had increased stock levels. The equilibrium feedback rule now entails a more prudent behavior. Actions are always too low in comparison with what the optimal trajectory would request. Along such a low-action trajectory, the tipping point is thought to be unlikely to have been passed yet; which in turn justifies adopting a more prudent behavior. This scenario gives foundations for the *Precautionary Principle* invoked by real-life decision-makers. Numerical simulations nevertheless suggest that the lack of information on past behavior does not entail a large welfare cost compared to the observable deviation scenario (less than 5%); softening concerns about the use of the *Precautionary Principle.*

Organization. Section 2 reviews the literature. Section 3 presents the model and the optimal solution. Section 5 contrasts how the latter can be implemented in a Stock-Markov game with observable deviations, but not under non-observable deviations, giving way to the Precautionary Principle. Section 8 briefly recaps our results and discusses possible extensions. Proofs are relegated into Appendices.

2. LITERATURE REVIEW

Irreversibility and the Precautionary Principle. Arrow and Fisher (1974), Henry (1974) and Freixas and Laffont (1984) were the first to show how a decision-maker should take more preventive stances when the consequences of irreversible choices are uncertain. This literature suggests that current abatements of greenhouse gaz emissions should be greater when more information will be available in the future (Chichilnisky and Heal, 1993; Beltratti, Chichilnisky and Heal, 1995; Kolstad, 1996 among others). Gollier, Jullien and Treich (2000) have built on this insight to give some economic content to the *Precautionary Principle.* They interpret the *Precautionary Principle* as the incentives to reduce actions below the level that would otherwise be optimal without uncertainty, when actions are taken before learning information. Asano (2010) has focused on the comparison of optimal environmental policies without and with ambiguity, showing that lack of confidence forces decision-makers to hasten policy adoption. In those models, decisions are always optimal although constrained by informational requirements[7] and information is exogenous whereas in many contexts in environmental economics, actions also determine information structures.[8]In contrast, we stress that beliefs on the state of the system are endogenous, determined by the history of past actions and what is known on their consequences. Relatedly, Salmi, Laiho and Murto (2019) study the trade-off faced by a decision-maker who must choose between acting now, which means taking a less informed decision but generating information that is useful in the sequel, and acting later, when being more informed. Greater actions accelerate learning.

On Tipping Points and Catastrophes. Catastrophic outcomes due to stock pollutants have been analyzed by Cropper (1976), Heal (1984) and Clarke and Reed (1994) among others. In those models, the probability of a catastrophe (be it irreversible or temporary) is increasing in the stock. Tsur and Zemel (1995) have investigated a problem of optimal resource extraction when extraction affects the probability that the resource becomes obsolete passed a certain threshold. When this threshold is unknown, the initial state affects the optimal path and there is less resource exploitation than under certainty. Sims and Finoff (2016) have studied how irreversibility in environmental damage and irreversibility in sunk cost investment interact in a model with tipping point uncertainty. Focusing on the optimal control of atmospheric pollution, Tsur and Zemel (1996) have shown how uncertainty on a tipping point introduces a multiplicity of possible equilibria. Tsur and Zemel (2021) have studied trajectories with state-dependent catastrophe thresholds. Contrary to us, these authors have focused on the case where the mere fact that the stock of pollutants has passed the tipping point is immediately learned by the decision-maker.[9] To capture the decision-maker's ignorance, another state variable reflecting his beliefs is introduced. This addition bears some resemblance to Crépin and Nævdal (2020)'s analysis. For the sake of realism, these authors have also added to state-dependent catastrophe models based on pollutants (or temperature) another state variable, the stress of the system, that triggers changes of regime only when it itself passes a threshold. Van der Ploeg (2014) has analyzed how uncertainty on tipping points may modify the design of an optimal dynamic path for carbon taxes. Lemoine and Traeger (2014) have investigated optimal policy in a context where decision-makers learn over the location of the tipping point over time from observing how the system

[7]This feature is shared by other models in the field like Immordino (2000) and Gonzales (2008).
[8]See Freixas and Laffont (1984) and Miller and Lad (1984).
[9]On this, see also Nævdal (2006).

responds. In the context of policies against global warming, they demonstrate that the possibility of regime switching significantly increases the optimal carbon tax. A similar empirical assessment has been obtained in Cai and Lontzek (2019). Finally, Liski and Salanié (2020) have also studied a model with unknown tipping points and uncertainty applied to climate change and pandemic crisis. These authors are particularly concerned with conditions ensuring whether actions are monotonic over time.

3. MODEL

PREFERENCES. A decision-maker, say DM, chooses actions over time. Time is continuous. Let $r > 0$ be the discount rate. Let $\mathbf{x} = (x(\tau))_{\tau \geq 0}$ (resp. $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$) denote an action plan (resp. the continuation of a plan from date t on).

Action $x(t)$ yields a flow payoff (net of the action cost) at date t worth $u(x(t))$. Although, we most often keep a general formulation, some of our results (optimal feedback rules and Hamilton-Bellman-Jacobi equations for value functions) are expressed in a crisper way by taking a quadratic specification, namely

$$u(x(t)) \equiv \zeta x(t) - \frac{x^2(t)}{2}$$

where $\zeta > 0$ is the marginal benefits of action (the consumption side) and $\frac{x^2(t)}{2}$ its cost (the production side). The set of feasible actions is $\mathcal{X} = [0, 2\zeta]$ so that flow payoff remains non-negative under all circumstances below.[10]

TECHNOLOGY AND CATASTROPHES. Actions put the environment at risk. A catastrophe may arise; an event that follows a Poisson process with a (non-homogeneous) rate $\theta(t)$. That rate depends on the stock $X(t) = \int_0^t x(\tau)d\tau$ of past actions before date t. More precisely, we postulate

$$\theta(t) = \theta_0 + \Delta \mathbb{1}_{\{X(t) > \overline{X}\}} \tag{3.1}$$

where $\overline{X}$ is a *tipping point.* Although it remains quite close to a homogeneous Poisson process, and indeed it is so before and after the tipping point, this specification features dependence on past actions. Indeed, when the stock of past actions $X(t)$ passes $\overline{X}$, the rate jumps from θ_0 to $\theta_1 > \theta_0$. Let $\Delta = \theta_1 - \theta_0 > 0$ measure this jump.

To capture its detrimental and irreversible impact, we assume that, if a catastrophe arises at date t, the flow payoff is no longer realized from that date on. A justification for this extreme assumption is that production may no longer be possible afterwards.[11]

A USEFUL BENCHMARK. We start with the simplest scenario where DM has no control over the arrival rate of a catastrophe, i.e., the case of a homogeneous Poisson process

[10] For simplicity, we assume that there is no flow damage $D(X(t))$ due to the stock of past pollutant but this possibility could be added to the model, although at the cost of unnecessary complications.

[11] A more general model would allow for an arbitrary number of catastrophes with possibly changes in the production/consumption structure following each of those events. This additional complexity would not add anything in terms of insights.

and we assume that the tipping point is at $\overline{X} = 0$, i.e., the tipping point is passed at the start. DM's expected payoff can thus be written as:

$$\int_0^{+\infty} e^{-\lambda_1 t} u(x(t))dt$$

where $\lambda_1 = r + \theta_1$ stands for the effective discount rate that applies with the possibility of a catastrophe. Since he cannot influence the arrival rate of the catastrophe, DM maximizes his intertemporal payoff by always choosing the *myopic action*

$$x^m(t) = \zeta \quad \forall t \geq 0.$$

For future reference, the myopic payoff once the tipping point has been passed writes as

$$\mathcal{V}_\infty = \frac{u(\zeta)}{\lambda_1}.$$

Of course, the same myopic action and payoff are obtained in any continuation; once it is known that the tipping point has been passed for sure.

4. BELIEFS, VALUE FUNCTION AND OPTIMAL TRAJECTORY

Suppose thus that DM does not know where the tipping point lies. Switching to the myopic optimum once the tipping point has been passed is no longer possible since DM remains ignorant on whether this event occurred or not.

4.1. *Beliefs*

Let denote by F the distribution of possible values for the tipping point and by f its (positive) density function. This distribution has a finite support $\left[0, \overline{X}\right]$ (i.e., $\overline{X} < +\infty$) and, for most of the paper, no mass point.

Consider a history of past actions $\mathbf{x}^t$ with no catastrophe up to date t and a stock reached at that date (starting from 0) given by $\hat{X}(t;0) = \int_0^t x(s)ds$. To evaluate DM's continuation payoff, we compute his posterior beliefs $f(\widetilde{X}|t, \mathbf{x}^t)d\widetilde{X}$ that the tipping point lies within the interval $\left[\widetilde{X}, \widetilde{X} + d\widetilde{X}\right]$ given that past history $\mathbf{x}^t$ at date t. This posterior density $f(\widetilde{X}|t, \mathbf{x}^t)$ should take into account that, if the tipping point lies at $\widetilde{X} \leq \hat{X}(t;0)$, the arrival rate has already jumped from θ_0 to θ_1 at an earlier date $T(\widetilde{X};0) \leq t$. If instead the tipping point is at $\widetilde{X} > \hat{X}(t;0)$, the arrival rate remains θ_0. A key variable to describe how the posterior density evolves is thus the probability of survival up to date t when the path of past actions is $\mathbf{x}^t$, namely

$$H(t, \mathbf{x}^t) = \int_0^{\hat{X}(t;0)} f(\widetilde{X}) e^{-\theta_0 T(\widetilde{X};0)} e^{-\theta_1 (t - T(\widetilde{X};0))} d\widetilde{X} + \int_{\hat{X}(t;0)}^{+\infty} f(\widetilde{X}) e^{-\theta_0 t} d\widetilde{X}. \tag{4.1}$$

After manipulations, we obtain:

$$H(t, \mathbf{x}^t) = e^{-\theta_0 t} \left(1 - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau;0)) e^{\Delta \tau} d\tau\right). \tag{4.2}$$ [12]

When the current stock $\hat{X}(\tau;0)$ is close to 0, the likelihood of having passed the tipping point is also close to 0. The survival probability is then nearly that obtained when the

[12] See the Proof of Lemma A.1 in the Appendix.

arrival rate of a catastrophe is known to be θ_0 for sure. As $\hat{X}(\tau;0)$ increases towards $\overline{X}$, it becomes more likely that the tipping point has been passed and the survival probability accordingly decreases. Of course, the shape of the distribution function F matters to evaluate this probability. As F puts more mass around the origin, it is more likely that the tipping point has been passed early on and the survival probability diminishes.

For future reference, let us define the *regime survival ratio* $\hat{Z}(t,\mathbf{x}^t)$ as

$$\hat{Z}(t,\mathbf{x}^t) = H(t,\mathbf{x}^t)e^{\theta_0 t} \quad \forall t \geq 0. \tag{4.3}$$

It is the ratio between the survival probability $H(t,\mathbf{x}^t)$ at date t following a history $\mathbf{x}^t$ and the survival probability $e^{-\theta_0 t}$ that would prevail had the tipping point never been passed.[13] This ratio actually reflects DM's beliefs on whether the tipping point has been passed or not. The faster the trajectory moves towards $\overline{X}$, the faster $\hat{Z}(t,\mathbf{x}^t) = 1-\Delta e^{-\Delta t}\int_0^t F(\hat{X}(\tau;0))e^{\Delta\tau}d\tau$ decreases. If the trajectory stays close to $X=0$, $\hat{Z}(t,\mathbf{x}^t)$ decreases very slowly. In other words, a higher value of $\hat{Z}(t,\mathbf{x}^t)$ can be viewed as reflecting greater optimism for DM. DM still thinks that the tipping point is ahead.

Running Example. Suppose that F has Dirac masses q at 0 and $1-q$ at $\overline{X}$. In other words, DM is uncertain whether the tipping point is passed right away or whether it will be later found at $\overline{X}$. For any $t>0$ and history $\mathbf{x}^t$ that has not yet reached $\overline{X}$, the probability of survival is a convex combination of exponential discounting:

$$H(t,\mathbf{x}^t) = qe^{-\theta_1 t} + (1-q)e^{-\theta_0 t}.$$

From this, it follows that the *regime survival ratio* before reaching $\overline{X}$ becomes

$$\hat{Z}(t,\mathbf{x}^t) = 1-q+qe^{-\Delta t}.$$

Note that $\hat{Z}(t,\mathbf{x}^t)$ is decreasing in t; capturing the fact that DM becomes more pessimistic as approaching the highest possible value of the tipping point $\overline{X}$. ■

4.2. *Value Function*

The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ is, by definition, DM's continuation payoff starting from date t onwards given the past history $\mathbf{x}^t$. This function is computed with the posterior density function $f(\widetilde{X}|t,\mathbf{x}^t)$ that the tipping point lies ahead of the current stock $X = \hat{X}(t;0)$ reached at date t (i.e., for $\widetilde{X} \geq \hat{X}(t;0)$) given that, following past history, no catastrophe has yet occurred. For $\tau \geq t$, the stock (denoted with a slight abuse of notations by $\hat{X}(\tau;X,t)$) will evolve according to the stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau\geq t}$. Lemma 1 provides a compact representation for this value function.

Lemma 1 *The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ satisfies*

$$\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,\hat{X}(\tau;X,t)=X+\int_t^\tau x(s)ds} \int_0^{+\infty} e^{-\int_0^\tau\left(\lambda_0-\frac{\frac{d\hat{Z}}{ds}(t+s,\mathbf{x}^{t+s})}{\hat{Z}(t+s,\mathbf{x}^{t+s})}\right)ds} u(x(t+\tau))d\tau \tag{4.4}$$

where $\lambda_0 = r+\theta_0$.

[13] Since the survival probability is bounded below by $e^{-\theta_1 t}$, the regime survival ratio itself lies within $(e^{-\Delta t},1]$.

The representation (4.4) of the value function suggests that the state of the system is best described by adding to the stock X a second state variable, the *regime survival ratio* Z that reflects beliefs. Two trajectories that have reached the same stock X with the same beliefs Z at a given date should have the same continuation. Instead, two trajectories that have reached the same stock but with different beliefs might be pursued differently. If the regime switch is thought as having been likely (Z small), DM will pursue with higher actions since he has less incentives to take a precautionary stance.

Representation of the Value Function. To complete the state of the system, we must thus add to the law of motion for the stock, namely

$$\dot{X}(\tau) = x(\tau), \tag{4.5}$$

the law of motion for the regime survival ratio.[14] Differentiating (4.3) and using (4.2) yields

$$\dot{Z}(\tau) = \Delta(1 - F(X(\tau)) - Z(\tau)). \tag{4.6}$$

Integrating (4.6) with the initial condition $Z(0) = Z$, we get the following expression for the regime survival ratio $Z(\tau)$:

$$Z(\tau) = \underbrace{1 - \Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds}_{\text{Memoryless Evolution}} \underbrace{-(1-Z)e^{-\Delta\tau}}_{\text{Pessimistic Stigma}}. \tag{4.7}$$

This expression highlights how the evolution of beliefs actually superposes two effects. Suppose that DM keeps no memory of what happened in the past. He is naively believing to start with $Z = 1$, only knowing about the current level of stock X and considering, from that point on, the ensuing trajectory $X(t)$ given by (4.5). The first term on the r.h.s. of (4.7) captures how such a naive DM would evaluate the consequences of pursuing this trajectory on future beliefs. Instead, whenever DM starts with some grain of pessimism inherited from past history, (i.e., starting with $Z < 1$) this (negative) *Pessimistic Stigma* is carried on in the future (although at a decreasing rate) and all the more so as Z is lower; an effect that is captured by the second term on the r.h.s. of (4.7).

Finally, (4.6) also implies that, once a trajectory $X(\tau)$ has reached the upper bound $\overline{X}$ at a date $\overline{T}$, the regime survival ratio evolves from then on as[15]

$$Z(\tau) = Z(\overline{T})e^{-\Delta(\tau - \overline{T})} \quad \forall \tau \geq \overline{T}. \tag{4.8}$$

Remark: For future reference, it is worth noticing that (4.6) together with the initial condition $Z(0) = Z$ imply that necessarily

$$Z(\tau) > 1 - F(X(\tau)) \quad \forall \tau \geq 0 \tag{4.9}$$

[14] Reed (1989) and Tsur and Zemel (1995) have developed dynamic optimization models which all have in common to use the survival probability as a state variable. The difference in our setting comes from the fact that this survival probability depends on where the trajectory lies in the distribution of possible tipping points.

[15] Once the stock level is beyond the support of F, the probability to be in the low-risk regime is 0.

and thus

$$\dot{Z}(\tau) < 0. \tag{4.10}$$

The first of those inequalities can be readily interpreted. Indeed, $1-F(X(\tau))$ is the probability that the tipping point lies above $X(\tau)$. Consider an alternative scenario where the fact of having passed the tipping point would be always immediately known (which also means that when not having crossed the tipping point yet, the rate of arrival of a catastrophe is known to be θ_0). The probability of survival conditional on not having crossed the tipping point yet at date τ along a path $X(\tau)$ would thus be $(1 - F(X(\tau)))e^{-\theta_0\tau}$. The regime survival ratio in that scenario would be $1 - F(X(\tau))$. Henceforth, (4.9) can be interpreted as saying that not knowing whether the tipping point has been passed, decision-makers somehow remain more optimistic. The second inequality (4.10) simply means that those decision-makers nevertheless become more pessimistic over time. ■

Using (4.4) and (4.8), we can now get a representation of the value function in terms of the bi-dimensional state variable (X, Z). Let accordingly define the value function $\mathcal{V}^e(X, Z)$ for $X \geq 0$ and any $Z \in (0, 1]$ as

$$\mathcal{V}^e(X, Z) = \sup_{\mathcal{A}} \int_0^{\overline{T}} e^{-\int_0^\tau \left(\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}\right)ds} u(x(\tau))d\tau + e^{-\int_0^{\overline{T}} \left(\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}\right)ds}\mathcal{V}_\infty. \tag{4.11}$$

where the set of admissible trajectories is

$$\mathcal{A} = \{\mathbf{x}, X(\cdot), Z(\cdot), \overline{T} \text{ s.t. (4.5), (4.6), } X(0) = X,\ X(\overline{T}) = \overline{X},\ Z(0) = Z\}.$$[16]

Starting from an arbitrary pair (X, Z), DM looks for an optimal arc that reaches $\overline{X}$ at some date $\overline{T}$. From that date on, DM knows for sure that the tipping point has been passed and chooses the myopic optimum. After having passed the tipping point at date $\overline{T}$, DM always chooses the myopic optimal action ζ and gets, from that date on, a discounted continuation payoff worth $\mathcal{V}_\infty$. In fact, the tipping point might have already been passed a long time ago but DM could not know it for sure before reaching $\overline{X}$.

The expression (4.11) showcases that, under uncertainty, the effective discount rate is time-dependent, namely

$$\lambda^e(\tau) \equiv \lambda_0 - \frac{\dot{Z}(\tau)}{Z(\tau)}.$$

Using the regime survival ratio as a state variable keeps track of this time-dependency. The choice of an action at any given date has no direct impact on how this implicit discount rate evolves since the law of motion (4.6) for beliefs does not depend on current action. Yet, because stock and beliefs evolve over time, this implicit discount rate keeps on changing and DM must take this into account to assess how future payoffs should be discounted. Specifically, DM is using $\lambda^e(\tau) \approx \lambda_0$ to discount future payoffs earlier on but, eventually, will switch to $\lambda^e(\tau) \approx \lambda_1$. The hazard rate $-\dot{Z}(\tau)/Z(\tau)$ measures how information contained in the fact that no catastrophe has yet happened is incorporated into this implicit discounting.

[16]We allow for the possibility that $\overline{T} = +\infty$. It turns out that the upper bound on its distribution is always reached in finite time for the optimal trajectory.

4.3. *Optimal Trajectory*

Next proposition presents some important properties of the value function $\mathcal{V}^e(X,Z)$ and the corresponding feedback rule.

PROPOSITION 1 *The value function $\mathcal{V}^e(X,Z)$ satisfies: the Hamilton-Bellman-Jacobi equation:*

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,Z) = -\zeta + \sqrt{2\lambda^e(X,Z)\mathcal{V}^e(X,Z) - 2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)} \ a.e. \tag{4.12}$$

where

$$\lambda^e(X,Z) = \lambda_0 - \frac{\Delta(1-F(X)-Z)}{Z} \tag{4.13}$$

together with the boundary conditions

$$\mathcal{V}^e(X,Z) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}, \forall Z \in (0,1]. \tag{4.14}$$

The optimal feedback rule is

$$\sigma^e(X,Z) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X,Z). \tag{4.15}$$

Before commenting on Proposition 1, it is useful to investigate a special case.

WHEN THE TIPPING POINT IS KNOWN. Suppose that the tipping point is known and located at $\overline{X} > 0$. Proposition 1 still applies provided one is ready to let $F(X) = 0$ for $X \in [0, \overline{X})$ with F having a mass point at $\overline{X}$. Inserting into (4.7) and using the fact that $Z = 0$ at $\tau = 0$, immediately yields $Z(\tau) = 1$ for all $\tau \geq 0$ in this scenario. Accordingly, we are now ready to further characterize the value function and the optimal feedback rule in this common knowledge scenario.

PROPOSITION 2 *The value function $\mathcal{V}^e(X,1)$ satisfies the following Hamilton-Bellman-Jacobi equation*

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,1) = -\zeta + \sqrt{2\lambda_0\mathcal{V}^e(X,1)}, \quad \forall X < \overline{X}.[17] \tag{4.16}$$

$\mathcal{V}^e(X,1)$ is decreasing and strictly concave for $X \in [0,\overline{X})$ with the boundary condition

$$\mathcal{V}^e(X,1) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{4.17}$$

The optimal feedback rule is

$$\sigma^e(X,1) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X,1). \tag{4.18}$$

Moreover, $\sigma^e(X,1)$ is decreasing in X for $X \in [0,\overline{X})$.

[17] At $X = \overline{X}$, this derivative is in fact a left-derivative but we use the same notation for simplicity.

Actions profile. The optimal action goes through two distinct phases. Before reaching the tipping point, actions have a long-lasting impact since they may contribute to passing the tipping point earlier on. Reducing those actions decreases the probability that a catastrophe arises earlier. The quantity $-\frac{\partial \mathcal{V}^e}{\partial X}(X,1)$ found on the r.-h.s. of (4.18) is in fact the Lagrange multiplier for the irreversibility constraint

$$\int_0^{\overline{T}} x(\tau)d\tau = \overline{X} - X. \tag{4.19}$$

As X increases towards $\overline{X}$, this irreversibility constraint becomes more demanding, and the value function decreases. Actions are below the myopic optimum to account for this *Irreversibility Effect.*

The optimal action decreases over time before the tipping point. All actions taken during this first phase contribute the same to the overall stock. Because of discounting, DM prefers to choose higher actions earlier on and lower ones when approaching the tipping point. Expressed in terms of the value function, this monotonicity means that $\mathcal{V}^e(X,1)$ is strictly concave over this first phase while it is flat once the tipping point has been passed. By then, DM knows that his actions will no longer have any impact on the arrival rate of a catastrophe and thus chooses the myopic optimum.

Tipping Point. Because actions are now lower than the myopic optimum over the first phase, the tipping point is reached at a date[18]

$$\overline{T}^k = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right)\frac{1 - e^{-\lambda_0 \overline{T}^k}}{\lambda_0} > \overline{T}^m = \frac{\overline{X}}{\zeta} \tag{4.20}$$

where $\overline{T}^m$ is the time necessary to reach the tipping point when acting myopically. Pushing a bit further in the future the date $\overline{T}^k$ at which the tipping point is reached by a small amount $d\overline{T}$ has costs and benefits. First, DM incurs a welfare loss for a longer period of time since, over the first phase, actions are below the myopic optimum. Second, increasing $\overline{T}^k$ maintains the arrival rate of a catastrophe at its low level θ_0 longer.

Positive Lower Bound on Actions. Because actions are decreasing before reaching the tipping point, we necessarily have

$$\zeta\sqrt{\frac{\lambda_0}{\lambda_1}} \le \sigma^e(X,1) \le \zeta \quad \forall X. \tag{4.21}$$

where the left-hand side above is the action at the end of the first phase.

Comments on Proposition 1. The comparison of the Hamilton-Bellman-Jacobi equations with and without uncertainty is instructive. The first difference between (4.16) and (4.12) is related to how future payoffs are discounted. Under uncertainty, the effective discount rate is now time-dependent. As a thought experiment, suppose that the evolution of the hazard rate $-\dot{Z}(\tau)/Z(\tau)$ were exogenously given. The implicit discount rate being low earlier on and higher later, the optimal solution would call for taking larger actions earlier on. In our model, this dynamics is endogenous. Current actions modify stock and beliefs and somewhat control the evolution of the hazard rate $-\dot{Z}(\tau)/Z(\tau)$.

[18] See the Appendix for details.

The second difference comes from a new term, not present under complete information, $-2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)$ on the r.-h.s. of (4.12). Less optimistic stances, i.e., lower values of Z are associated with lower continuation values (i.e., $\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)<0$). Along the optimal trajectory, this new term is negative.[19] Being less optimistic and thinking that the tipping point has already been passed, DM certainly chooses to increase actions.

Finally, the comparison of the feedback rule (4.15) with its complete information counterpart (4.18) shows that the term $\frac{\partial \mathcal{V}^e}{\partial X}(X,Z)$ can again be interpreted as an opportunity cost of irreversibility. This cost now depends on beliefs. The consequences of such beliefs on actions can be further illustrated in the framework of our example.

Running Example (Continued). Although $\mathcal{V}^e(X,Z)$ cannot be expressed in closed form for $q>0$, both the profile of optimal actions $\mathbf{x}^e$ along the trajectory and the delay $\overline{T}^e$ till reaching the tipping point, can be solved explicitly.

Proposition 3 *Suppose that F has Dirac masses q at 0 and $1-q$ at $\overline{X}$. The optimal trajectory starting from $X=0$ and $Z=1$ has the following features.*

- *The date $\overline{T}^e$ at which $\overline{X}$ is reached solves*

$$\overline{T}^e=\overline{T}^m+\left(1-\sqrt{\frac{\lambda_0-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\int_0^{\overline{T}^e}\frac{Z(\overline{T}^e)}{Z(\tau)}e^{-\lambda_0(\overline{T}^e-\tau)}d\tau>\overline{T}^m \tag{4.22}$$

where the regime survival ratio is

$$Z(\tau)=1-q+qe^{-\Delta\tau}\quad \forall\tau\in[0,\overline{T}^e]. \tag{4.23}$$

- *The optimal action is decreasing over $\tau\in[0,\overline{T}^e)$ and equal to the myopic optimum thereafter:*

$$x^e(\tau)=\begin{cases}\zeta\left(1-e^{-\lambda_0(\overline{T}^e-\tau)}\frac{Z(\overline{T}^e)}{Z(\tau)}\left(1-\sqrt{\frac{\lambda_0-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\right)<\zeta & \text{for } \tau\in[0,\overline{T}^e),\\ \zeta & \text{for } \tau\geq\overline{T}^e.\end{cases} \tag{4.24}$$

The *Irreversibility Effect* is again at play as long as the highest possible values of the tipping point has not been passed. Actions remain below the myopic optimum over that first phase. Yet, actions are higher under uncertainty. This result is illustrated by observing that the last action before passing $\overline{X}$ has now been raised towards the myopic solution in comparison with the scenario where the tipping point isat $\overline{X}$ for sure:

$$x^e(\overline{T}^{e-})=\zeta\sqrt{\frac{\lambda_0-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}>\zeta\sqrt{\frac{\lambda_0}{\lambda_1}}. \tag{4.25}$$

Under uncertainty, the date at which the stock reaches $\overline{X}$ comes earlier on:

$$\overline{T}^e<\overline{T}^k.$$

Intuitively, there is now a chance that the tipping point has already been passed before, so that optimal actions are closer to the myopic optimum. It can also be readily checked that as q goes to 0 (resp. 1), $\overline{T}^e$ converges towards $\overline{T}^k$ (resp. $\overline{T}^m$). ■

[19]Indeed, we have $-\dot{Z}(\tau)/Z(\tau)=-\frac{\Delta(1-F(X(\tau))-Z(\tau))}{Z(\tau)}>0$.

5. *STOCK-MARKOV* EQUILIBRIUM WITH OBSERVABLE DEVIATIONS

The value function $\mathcal{V}^e(X, Z)$ is a mere technical device to use dynamic programming techniques and compute a feedback rule $\sigma^e(X, Z)$ that guides behavior along the optimal trajectory. There are two ways of thinking about this device. First, this feedback rule may be viewed as a machine that determines actions that a planner would take at each point in time in response to the evolution of stock and beliefs. Second, and it is a direct consequence of the *Principle of Dynamic Programming*, such feedback rule can be viewed as a Perfect-Markov equilibrium strategy among various selves of this decision-maker. In this non-cooperative scenario, selves acting at different points in time have only a limited ability to commit to an action over an infinitesimal period of time. They adopt Markov-strategies based on the state variable (X, Z). Because those selves are endowed with the same objectives and the same information than what a long-lived planner would have, their choice of the best action obviously replicates that of this planner.

Hereafter, we instead ask whether a more parsimonious decentralization of an optimal trajectory is also reached as a Perfect-Markov equilibrium if those selves were to adopt less complete *Stock-Markov* feedback rules that only depend on the stock X. Our motivation for looking at such a restriction on equilibrium strategies is that, in practice, only the stock of pollutants in the atmosphere can be easily verified and this stock might not be a sufficient statistics to form correct beliefs on whether the tipping point has likely been passed or not. Even though selves are still endowed with the same objectives, this restriction on feasible strategies may bite and affect the implemented action plan. We will show below that the extent to which it is so depends on whether impulse deviations are observable or not (Section 6 below). First, instead of having a single decision-maker choosing actions, the trajectory is viewed as the outcome of a game with different selves acting at different points in time. Those selves choose actions that prevail only for an infinitesimal period of time; a so called *impulse deviation*. Second, we consider that, when acting, those selves might have only limited information on the consequences of past acts (Section 6). At a *Stock-Markov Equilibrium* (thereafter SME), those selves adopt a feedback rule based only on stock.

Those modeling assumptions certainly echo the framework in which the *Precautionary Principle* is invoked. First, the concern that current actions may negatively impact future generations is captured by having different decision-makers, each endowed with the discounted flow of future payoffs, acting at different points in time. Second, the fact that current selves have only limited information on the consequences of past acts is a necessary ingredient to assess whether equilibrium actions might be more cautious under those circumstances. When taken in tandem, those assumptions allow us to assess whether an optimal trajectory can be decentralized as a non-cooperative equilibrium among selves and if not, the nature of the distortion.

5.1. *Setting the Stage*

We now consider a game in continuous time among selves of the decision-maker who act at different points in time. At any point in time τ, the current self DM_τ has limited commitment ability. He can only choose an action $x(\tau)$ over an interval of infinitesimal length of time $[\tau, \tau + \varepsilon]$. DM_τ's objective is to maximize intertemporal welfare from that date on given whatever information is available to him at date τ. In the first subsection, we suppose that DM_τ can observe whatever actions may have been undertaken by all his predecessors $DM_{\tau'}$ for $\tau' < \tau$ both on and off equilibrium path. Subsection 6 below will entertain the opposite scenario where past actions remain unknown.

Stock-Markov equilibria are supported by *Stock-Markov* feedback rules. At any such equilibrium, the self DM_τ in charge at date τ sticks to the strategy $\sigma^o(X)$ when the stock has reached level X because he expects future selves to abide to that rule as well following the subsequent evolution of the system.[20]

Along any such *Stock-Markov* trajectory, the stock $X^o(\tau; X)$ evolves as

$$\frac{\partial X^o}{\partial \tau}(\tau; X) = \sigma^o(X^o(\tau; X)) \text{ with } X^o(0; X) = X. \tag{5.1}$$

The various selves should also be able to reconstruct the regime survival ratio that applies, along the equilibrium path, for each possible level of the stock and, by that means, correctly infer how to discount future payoffs. Let denote by $Z^o(X)$ such function. From (4.6), the regime survival ratio $Z(\tau; X)$, that starts from value $Z^o(X)$ at date 0 and that is consistent with the *Stock-Markov* feedback rule $\sigma^o(X)$ from that date on evolves as

$$\frac{\partial Z}{\partial \tau}(\tau; X) = \Delta(1 - F(X^o(\tau; X)) - Z(\tau; X)) \text{ with } Z(0; X) = Z^o(X). \tag{5.2}$$

Since conjectures on how the regime survival ratio evolves along the trajectory are correct on the equilibrium path, we must also have

$$Z(\tau; X) = Z^o(X^o(\tau; X)) \quad \forall \tau \geq 0, X \geq 0. \tag{5.3}$$

Taken together, those conditions dictate how the regime survival ratio evolves with the current stock along the trajectory. Differentiating (5.3) with respect to τ yields

$$\sigma^o(X)\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \tag{5.4}$$

with the initial condition

$$Z^o(0) = 1. \tag{5.5}$$

We may now define a *Stock-Markov value function* $\mathcal{V}^o(X)$, i.e., the intertemporal payoff along such a *Stock-Markov* trajectory, as

$$\mathcal{V}^o(X) = \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \sigma^o(X^o(s;X))\frac{\dot{Z}^o(X^o(s;X))}{Z^o(X^o(s;X))}\right)ds} u(\sigma^o(X^o(\tau; X)))d\tau. \tag{5.6}$$

This definition showcases how future payoffs are discounted at a rate

$$\lambda_0 - \sigma^o(X^o(s; X))\frac{\dot{Z}^o(X^o(s; X))}{Z^o(X^o(s; X))}$$

that depends on the regime survival ratio along the *Stock-Markov* trajectory.

[20]Of course, a *Stock-Markov* feedback rule should specify that $\sigma^o(X) = \zeta$ for $X \geq \overline{X}$ but, in order to save on notations, this expression of the continuation will be kept implicit in what follows.

For future reference, we define the intertemporal payoff once the tipping point has been passed for sure but, being ignorant of that event, all future selves still rely on the feedback rule $\sigma^o(X)$ to choose actions, as

$$\varphi^o(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X)))d\tau. \tag{5.7}$$

Impulse Deviations. To express the equilibrium requirement that sticking to the feedback rule $\sigma^o(X)$ is optimal at any point along the trajectory, we follow an approach that was developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015), Ekeland and Lazrak (2006, 2008, 2010) and Auster, Che and Mierendorff (2023). These authors have analyzed dynamic decision-making models with time-inconsistency problems. To model non-cooperative action choices by various decision-makers (or selves of the same decision-maker), the notion of perfect-Markov equilibrium was imported into a continuous time setting. The idea is to look at the benefits of deviating from the feedback rule for periods of commitment which are of arbitrarily small length; deriving from there conditions for the sub-optimality of such deviations and thus properties of the equilibrium feedback rule.

To this end, consider a possible deviation that would consist for the current self in committing to an action x for a period of length ε, reaching a stock level $X + x\varepsilon$, before subsequent selves jumping back to the feedback rule σ^o. For such an *impulse deviation*, actions are thus

$$y(x, \varepsilon, \tau; X) = \begin{cases} x & \text{if } \tau \in [0, \varepsilon], \\ \sigma^o(\hat{X}(x, \varepsilon, \tau; X)) & \text{if } \tau > \varepsilon \end{cases} \tag{5.8}$$

while the whole stock trajectory is modified as

$$\hat{X}(x, \varepsilon, \tau; X) = \begin{cases} X + x\tau & \text{if } \tau \in [0, \varepsilon], \\ X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(\hat{X}(x, \varepsilon, s; X))ds & \text{if } \tau \geq \varepsilon. \end{cases} \tag{5.9}$$

By adopting such impulse deviation, the regime survival ratio also changes as

$$\hat{Z}(x, \varepsilon, \tau; X) = 1 - \Delta e^{-\Delta \tau} \int_0^\tau F(\hat{X}(x, \varepsilon, s; X))e^{\Delta s} ds - (1 - Z^o(X))e^{-\Delta \tau}. \tag{5.10}$$

From this, we may define DM's deviation payoff $\hat{\mathcal{V}}(x, \varepsilon; X)$ as

$$\hat{\mathcal{V}}(x, \varepsilon; X) = \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \frac{\frac{\partial \hat{Z}}{\partial s}(x,\varepsilon,s;X)}{\hat{Z}(x,\varepsilon,s;X)}\right) ds} u(y(x, \varepsilon, \tau; X))d\tau. \tag{5.11}$$

That all future selves are able to observe any impulse deviation that the current decision-maker may entertain allows those selves to reconstruct the evolution of beliefs as expressed in (5.10). When considering the consequences of any impulse deviation, the current decision-maker should thus assess those consequences on his intertemporal payoff by applying the implicit discounting that follows from the evolution of beliefs so induced. This inference is clear in the expression of the continuation payoff on the right-hand side of (5.11).

DEFINITION 1 *A triplet $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$ is a SME with observable impulse deviations if the following conditions hold.*

1. *$\mathcal{V}^o(X)$ as defined by (5.6) cannot be improved upon by any impulse deviation of the form (5.8)-(5.9) for ε made arbitrarily small:*

$$\mathcal{V}^o(X) = \max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}(x,\varepsilon;X). \tag{5.12}$$

2. *$\sigma^o(X)$ is optimal for ε made arbitrarily small:*

$$\sigma^o(X) \in \arg\max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}(x,\varepsilon;X). \tag{5.13}$$

3. *$Z^o(X)$ is consistent with the feedback rule $\sigma^o(X)$ and satisfies (5.4)-(5.5).*

Item 1. requires to approximate the deviation payoff $\hat{\mathcal{V}}(x,\varepsilon;X)$ to the first order in ε and look for the optimal action that maximizes such approximation; an optimality condition that is expressed in Item 2. Those two steps are familiar from applying the *Principle of Dynamic Programming* in contexts with time-consistent plans. Item 3. follows from the consistency condition (5.3) which states that the optimal evolution of beliefs is dictated by the *Stock-Markov* feedback rule. This step is more novel. Of course, the evolution of the survival ratio should be consistent with this feedback rule.

PROPERTIES OF $(\mathcal{V}^o(X), \sigma^o(X))$. Developing the equilibrium conditions in Definition 1 yields important properties.

PROPOSITION 4 *At any (continuously differentiable) SME, with observable impulse deviations, the Stock-Markov value function $\mathcal{V}^o(X)$ satisfies the following functional equation*

$$\dot{\mathcal{V}}^o(X) = -\zeta - \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X) + \sqrt{2\lambda_0\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2} \quad \forall X \in [0,\overline{X}) \tag{5.14}$$

together with the boundary condition

$$\mathcal{V}^o(X) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{5.15}$$

The corresponding Stock-Markov feedback rule writes as

$$\sigma^o(X) = \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}(\mathcal{V}^o(X) - \varphi^o(X)). \tag{5.16}$$

The formula for the feedback rule in (5.16) bears some resemblance with its counterpart (4.18) that was found under complete information. To understand the changes, it is useful to come back on the expression of the *Stock-Markov* value function (5.6). Starting from a current stock X with current beliefs $Z^o(X)$ on the equilibrium path, consider an impulse deviation consisting in increasing by a marginal amount dx the current action $\sigma^o(X)$ over an interval of infinitesimal length ε. Since the current stock increases by $dX = \varepsilon dx$, such impulse deviation reduces the *Stock-Markov* value function by

$$-\dot{\mathcal{V}}^o(X)\varepsilon dx. \tag{5.17}$$

This impact can be decomposed into three different components. First, this impulse deviation yields a marginal benefit on current payoff over the infinitesimal interval worth

$$(\zeta - \sigma^o(X))\varepsilon dx. \tag{5.18}$$

Second, this impulse deviation also increases the implicit discount rate that applies to future payoffs by

$$\frac{\dot{Z}^o(X)}{Z^o(X)}\varepsilon dx < 0.$$

The corresponding impact on continuation payoff is thus a reduction in continuation payoff worth

$$\frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X)\varepsilon dx < 0. \tag{5.19}$$

This effect decreases current action. Importantly, it is entirely due to the induced change in stock. It takes as given the evolution of beliefs and would be also present if the rate $\frac{\dot{Z}^o(X)}{Z^o(X)}$ at which the survival ratio evolves was taken as given. This will be the case in Section 6 below here we investigate the scenario of non-observable impulse deviations.

Because here it is observable by future selves, an impulse deviation has nevertheless also a long-lasting effect on beliefs as highlighted by formula (5.10). A marginal increase in the stock worth εdx makes it more likely that the tipping point has been passed within the infinitesimal interval where this impulse deviation applies. It brings an extra grain of pessimism over the whole future trajectory. From (5.10), this deviation indeed impacts the *Pessimistic Stigma* by a term which, at a date τ beyond the impulse deviation, is

$$\dot{Z}^o(X)e^{-\Delta\tau}\varepsilon dx < 0.$$

Passed the tipping point, payoffs would be discounted at rate λ_1 if this event was observed leading to an intertemporal gain worth $\varphi^o(X)$. The benefit of believing that the tipping point is more likely to have been passed following this impulse deviation is thus

$$-\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\varepsilon dx > 0. \tag{5.20}$$

Since a more pessimistic decision-maker chooses higher actions, this last effect increases current action.

Gathering (5.17), (5.18), (5.19) and (5.20) above finally yields Condition (5.16) which characterizes the optimal feed-back rule.

Reciprocally, a triplet $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$ that satisfies (5.14), (5.15), (5.16) and the consistency requirements (5.4)-(5.5) forms a *SME*. This point is exploited in Proposition 5 below to show that an optimal arc can be implemented as a *SME*.

Remark. Consider the alternative scenario where DM remains ignorant on where the tipping point lies but, thanks to hard scientific evidence, immediately learns it upon passing it.[21] DM thus knows that his payoffs should be discounted at rate λ_0 as long as he has not yet learned having passed the tipping point. The dynamics of the system is thus fully summarized by the stock X that can be used as the sole state variable. Observe also that the probability of not having yet switched regime is then $1 - F(X)$ and that, once the tipping point has been passed, the myopic action is chosen which yields a continuation payoff $\mathcal{V}_\infty$. Denoting by $\mathcal{V}^u(X)$ the value function conditionally on not having yet learned that the tipping point has been passed, we may adapt our previous analysis to express this value function as

$$(5.21)\quad \mathcal{V}^u(X) = \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0+\sigma^u(X^u(s;X))\frac{f(X^u(s;X))}{1-F(X^u(s;X))}\right)ds} u(\sigma^u(X^u(\tau;X)))d\tau$$

and get the optimal feedback rule $\sigma^u(X)$ as

$$(5.22)\quad \sigma^u(X) = \zeta + \dot{\mathcal{V}}^u(X) - \frac{f(X)}{1-F(X)}(\mathcal{V}^u(X) - \mathcal{V}_\infty).$$

This formula bears some obvious resemblance with (5.16). Upon learning that he has passed the tipping point, an event whose hazard rate is $\frac{f(X)}{1-F(X)}$, DM knows for sure that the continuation payoff drops from $\mathcal{V}^u(X)$ to $\mathcal{V}_\infty$. In order to postpone this drop, DM reduces current actions. ■

Implementation of the Optimal Trajectory. The evolution of beliefs along a *SME* is determined by the feedback rule on path. If DM expects future selves to stick to a *Stock-Markov* rule that implements the optimal action profile, he also expects beliefs to be modified as expected at the optimum. Hence, when considering the possible benefits of an observable impulse deviation, there is nothing that distinguishes the current self when he is playing the SME defined in Proposition 5 from a long-lived planner who would be considering the impact of a marginal change of action on the future stream of payoffs. Because impulse deviations are observable, future selves will modify beliefs as thus planner would also do and will accordingly choose the same actions profile.

Proposition 5 *Suppose that impulse deviations are observable, an optimal path can be implemented as a SME,*[22] $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$*, such that*

$$(5.23)\quad \mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X)) \text{ and } \sigma^o(X) = \sigma^e(X, Z^o(X)) \quad \forall X$$

with $Z^o(X)$ *being consistent with the feedback rule* $\sigma^o(X)$ *and satisfying (5.4)-(5.5).*

Running Example (Continued). Consider the trajectory starting from $X = 0$ and

[21] This scenario is analyzed in Tsur and Zemel (1996, 2021) among others and is isomorphic to Loury (1978)'s analysis of how to exploit a resource with unknown reserve. In that model, when DM has reached the limits of the resource stock, he immediately knows it and stops consuming from then on.

[22] The difficulty in directly proving existence of a *SME* comes from the fact that the differential equation (5.14) for $\mathcal{V}^o(X)$ depends on DM's payoff $\varphi^o(X)$ in case the tipping point has been passed which itself depends on the *Stock-Markov* feedback rule computed over the whole future trajectory. Local existence results are of little help given that non-local property. Proposition 5 overcomes this difficulty, in proving the existence of a *SME* indirectly from the existence of an optimal path.

$Z = 1$. From the expression of the optimal action (4.24), the stock evolves as

$$(5.24)\quad X^e(\tau) = \begin{cases} \zeta\left(\tau - \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\int_0^\tau \frac{Z(\overline{T}^e)}{Z(s)} e^{-\lambda_0(\overline{T}^e - s)} ds\right) & \text{for } \tau \in [0, \overline{T}^e), \\ \overline{X} + \zeta(\tau - \overline{T}^e) & \text{for } \tau \geq \overline{T}^e. \end{cases}$$

Together with (4.23), this expression allows us to recover an almost closed form for $X^o(Z)$ (the inverse function of $Z^o(X)$) for $Z \in [1 - q + qe^{-\Delta \overline{T}^e}, 1]$ as

(5.25)

$$X^o(Z) = \zeta\left(-\frac{1}{\Delta} ln\left(1 + \frac{Z-1}{q}\right) - \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\int_0^{-\frac{1}{\Delta} ln\left(1 + \frac{Z-1}{q}\right)} \frac{Z(\overline{T}^e)}{Z(s)} e^{-\lambda_0(\overline{T}^e - s)} ds\right).$$

It can be readily verified that

$$\dot{X}^o(Z(\overline{T}^e)) = \frac{\zeta e^{\Delta \overline{T}^e}}{q\Delta}\sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}.$$

We thus get $\lim_{q\to 1} \dot{X}^o(Z(\overline{T}^e)) = 0$ or, equivalently, $\lim_{q\to 1} \dot{Z}^o(\overline{X}^-) = -\infty$. Intuitively, when q is close to one, the function $Z^o(X)$ also remains close to one for most values of X, only decreasing very quickly towards $1 - q + qe^{-\Delta \overline{T}^m}$ when X comes close to $\overline{X}$. Finally, the optimal action at $\overline{X}^-$, namely $\sigma^o(\overline{X}^-) = x^e(\overline{T}^e)$ (which is expressed in (4.25)) indeed converges towards the lowest bound $\zeta\sqrt{\frac{\lambda_0}{\lambda_1}}$ as q goes to zero. ■

6. *STOCK-MARKOV* EQUILIBRIA WITH NON-OBSERVABLE DEVIATIONS

We now consider a scenario where the self DM_τ in charge over a period of infinitesimal length around date τ does not observe any impulse deviations that his predecessors $DM_{\tau'}$ for $\tau' < \tau$ may have entertained. Only the current level of the stock $X = X(\tau)$ remains observable for DM_τ. In practice, the consequences of an action at a given point in time may only be detected after a lag. Hereafter, we will take the polar view that the lag for detecting any impulse deviation is infinite. One possible justification is that scientific knowledge might not be sufficiently advanced to assess those consequences right away. An alternative explanation is that the selves might have bounded rationality and limited ability to process information. Accordingly, we need to slightly modify the notion of *SME* to account for the non-observability of impulse deviations.

6.1. *Setting the Stage*

In any such *SME*, all selves conjecture that the feedback rule $\sigma^{no}(X)$ is adopted. Accordingly, they all believe that the regime survival ratio evolves according to

$$(6.1)\quad \sigma^{no}(X)\dot{Z}^{no}(X) = \Delta(1 - F(X) - Z^{no}(X)) \quad \forall X \geq 0^{23}$$

with the initial condition

$$(6.2)\quad Z^{no}(0) = 1.$$

Because necessarily $\sigma^{no}(X) = \zeta$ for $X > \overline{X}$, (6.1) immediately imply

$$Z^{no}(X) = Z^{no}(\overline{X})e^{-\frac{\Delta}{\zeta}(X-\overline{X})} \quad \forall X > \overline{X}. \tag{6.3}$$

For any stock $X \leq \overline{X}$, we may now define the *Stock-Markov value function with non-observable deviations* $\mathcal{V}^{no}(X)$ along such *SME* as:

$$\mathcal{V}^{no}(X) = \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \sigma^{no}(X^{no}(s;X))\frac{\dot{Z}^{no}(X^{no}(s;X))}{Z^{no}(X^{no}(s;X))}\right)ds} u(\sigma^{no}(X^{no}(\tau;X)))d\tau. \tag{6.4}$$

6.2. *Impulse Deviations*

An impulse deviation again entails a modification of the action profile as specified in (5.8) and an ensuing evolution of the stock as in (5.9). Because impulse deviations are now not observable, a deviation by DM_τ has no impact on the degree of pessimism that his followers $DM_{\tau'}$, for $\tau' > \tau$ adopt. They still believe that the regime survival ratio evolves on path as specified in (6.1) and (6.2). Of course, an impulse deviation made earlier on modifies the current stock and affects where the regime survival ratio lies along this trajectory. This point is made clear in the following expression of the payoff for such a deviation:

$$\hat{\mathcal{V}}^{no}(x,\varepsilon;X) = \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \frac{\partial \hat{X}}{\partial s}(x,\varepsilon,s;X)\frac{\dot{Z}^{no}(\hat{X}(x,\varepsilon,s;X))}{Z^{no}(\hat{X}(x,\varepsilon,s;X))}\right)ds} u(y(x,\varepsilon,\tau;X))d\tau. \tag{6.5}$$

From there, we deduce the following definition.

Definition 2 *A triplet $(\mathcal{V}^{no}(X), \sigma^{no}(X), Z^{no}(X))$ is a SME with non-observable deviations if the following conditions hold.*

1. *$\mathcal{V}^{no}(X)$ as defined by (6.4) cannot be improved upon by any impulse deviation of the form (5.8)-(5.9) for ε made arbitrarily small:*

$$\mathcal{V}^{no}(X) = \max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}^{no}(x,\varepsilon;X). \tag{6.6}$$

2. *$\sigma^{no}(X)$ is optimal for ε made arbitrarily small:*

$$\sigma^{o}(X) \in \arg\max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}^{no}(x,\varepsilon;X). \tag{6.7}$$

3. *$Z^{no}(X)$ is consistent with the feedback rule $\sigma^{no}(X)$ and satisfies (6.1)-(6.2).*

This definition looks very much alike Definition 1. Both definitions require first, that impulse deviations should not improve payoffs locally (Item 1.) and second, that the evolution of the regime survival ratio should be consistent with the feedback rule (Item 3.). The key difference between Definitions 1 and 2 comes from the fact that deviation payoffs are written differently. With observable deviations, continuation payoffs following an impulse deviation are modified to account for how the regime survival ratio carries over changes in the *Pessimistic Stigma*. With non-observable deviations, subsequent decision-makers are more naive. The sole impact of an impulse deviation on continuation payoff is to change the level of stock and thus the implicit discount rate that applies to how they compute future payoffs. Decision-makers take the evolution of beliefs as fixed when considering a deviation.

REMARKS. Two implicit assumptions are made. First, each self only knows the current level of stock when acting. Suppose instead, that he would have known for how long the project has been run, or at which point in time he is acting. Conjecturing that previous selves have abided to the *Stock-Markov* feedback rule that prevails at equilibrium and comparing with the current stock he is observing would allow this self to detect that (at least) one deviation has taken place earlier on, even if he might not be able to infer at which date it was. Assuming that only the current stock is observed avoids such inference and accordingly simplifies the analysis. This assumption is akin to suppose that selves are naive and have limited memory. They can just keep track of the level of stock but cannot figure out the precise actions path that induces such stock. Alternatively, it could be that the initial level of stock remains unknown so that correct inferences on whether a deviation took place are not feasible either.

Second, because impulse deviations are non-observable, all selves believe that the regime survival ratio still evolves as on path, i.e., as in (6.1). Instead, when deviating at date τ, DM_τ knows that the correct evolution of beliefs is given by (5.10). This difference *a priori* implies that, beyond the commitment period whose length is infinitesimal, the discounted intertemporal streams of utilities evaluated with DM_τ's beliefs and that of his future selves $DM_{\tau'}$ for $\tau' > \tau$ differ. To fix this issue, focus on the main consequences of non-observability in the simpler scenario and again simplify the analysis, we assume that DM_τ cares about the intertemporal payoff of his subsequent selves; thus considering their own beliefs when evaluating his future payoffs. From this, we may thus define DM's deviation payoff $\hat{\mathcal{V}}(x, \varepsilon; X)$ as in (6.5). ■

6.3. *Equilibrium Properties*

Next proposition echoes our findings in Proposition 4 but now considering a scenario with non-observable deviations.

PROPOSITION 6 *At any (continuously differentiable) SME with non-observable impulse deviations, the Stock-Markov value function $\mathcal{V}^{no}(X)$ satisfies the following Hamilton-Bellman-Jacobi equation*

$$\dot{\mathcal{V}}^{no}(X) = -\zeta - \frac{\dot{Z}^{no}(X)}{Z^{no}(X)}\mathcal{V}^{no}(X) + \sqrt{2\lambda_0 \mathcal{V}^{no}(X)} \quad \forall X \in [0, \overline{X}) \tag{6.8}$$

together with the boundary condition

$$\mathcal{V}^{no}(X) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{6.9}$$

The Stock-Markov feedback rule is

$$\sigma^{no}(X) = \zeta + \dot{\mathcal{V}}^{no}(X) + \frac{\dot{Z}^{no}(X)}{Z^{no}(X)}\mathcal{V}^{no}(X) \quad \forall X \in [0, \overline{X}). \tag{6.10}$$

The feedback rule with non-observable deviations (6.10) is much like its counterpart (5.16) found when those deviations are observable. Yet, the term (5.20) is missing. To explain this omission, consider again increasing by a small amount dx the current action $\sigma^{no}(X)$ over an interval of infinitesimal length ε, starting from a current stock X with current beliefs $Z^{no}(X)$. If this impulse deviation is non-observable, future selves, when choosing their own actions, only consider its impact on the observable stock which has increased by εdx. The comparison with observable deviations is thus straightforward.

First, this non-observable impulse deviation still impacts current payoff because the feedback rule $\sigma^{no}(X)$ requires a change in action at this new level of stock. This term is again given by (5.17). Second, this impulse deviation also increases the implicit discount rate; a term which is still captured by (5.18). Yet, with a non-observable deviation, the regime survival ratio $Z^{no}(X)$ is taken as given over the whole trajectory. Had such a deviation been observable, DM_τ instead would have known that increasing current action also means that future beliefs carry on some *Pessimistic Stigma* and this pessimism makes it more attractive for future selves, $DM_{\tau'}$ for $\tau' > \tau$ who think that the tipping point may have been passed, to further increase actions later on. With a non-observable deviation, this motive for raising actions disappears and actions remain low.

At equilibrium, the feedback rule now calls for excessively low actions in comparison with the optimal trajectory. Indeed, in any SME with observable deviations, we have

$$\sigma^o(X) > \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X).$$

With low actions early on, the conjectured evolution of beliefs remains quite optimistic. Each self thinks that the tipping point remains unlikely to have been already passed when he acts and, in response, adopts a prudent behavior. This prudent behavior is of course excessive in comparison with the optimal trajectory. Yet, it is self-fulfilling.

Running Example (Continued). The trajectory under a *SME* with non-observable impulse deviations can again be computed in (almost) closed form.

Proposition 7 *Suppose that F has Dirac masses q at 0 and $1-q$ at $\overline{X}$. The trajectory under a SME with non-observable impulse deviations starting from $X = 0$ and $Z = 1$ has the following features.*

- *The date $\overline{T}^{no} > \overline{T}^k$ at which $\overline{X}$ is reached solves*

(6.11)

$$\overline{T}^m = \sqrt{\frac{\lambda_0}{\lambda_1}}\left(\int_0^{\overline{T}^{no}} \sqrt{\frac{Z(T^{no})}{Z(\tau)}} e^{-\lambda_0(\overline{T}^{no}-\tau)} d\tau\right) + \lambda_0 \int_0^{\overline{T}^{no}} \left(\int_\tau^{\overline{T}^{no}} \sqrt{\frac{Z(s)}{Z(\tau)}} e^{\lambda_0(\tau-s)} ds\right) d\tau$$

where $Z(\tau)$ is given by

$$(6.12) \qquad Z(\tau) = 1 - q + qe^{-\Delta\tau} \quad \forall \tau \in [0, \overline{T}^{no}].$$

- *The action $x^{no}(\tau)$ satisfies*

(6.13)

$$x^{no}(\tau) = \begin{cases} \zeta \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}\left(\sqrt{Z(T^{no})}e^{-\lambda_0\overline{T}^{no}}\sqrt{\frac{\lambda_0}{\lambda_1}} + \lambda_0\int_\tau^{\overline{T}^{no}}\sqrt{Z(s)}e^{-\lambda_0 s}ds\right) < \zeta & \text{for } \tau \in [0, \overline{T}^{no}), \\ \zeta & \text{for } \tau \geq \overline{T}^{no}. \end{cases}$$

To illustrate the tendency for choosing low actions when impulse deviations are non-observable, observe that the last action before jumping to the myopic optimum is always lower than with observable deviations:

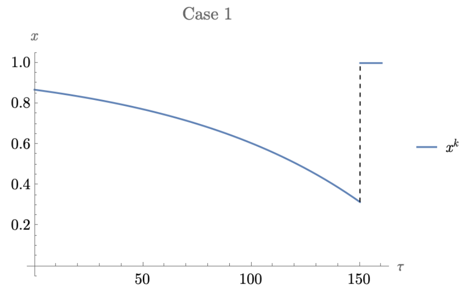
$$x^{no}(\overline{T}^{no}) = \sqrt{\frac{\lambda_0}{\lambda_1}} < x^o(\overline{T}^e) = x^e(\overline{T}^e) = \sqrt{\frac{\lambda_0}{\lambda_1} + \frac{q\Delta e^{-\Delta \overline{T}^e}}{1-q+qe^{-\Delta \overline{T}^e}}}.$$

■

7. NUMERICAL SIMULATIONS

The debate on the relevance of the *Precautionary Principle* would really matter if the trajectories with and without observability of deviations were significantly different in terms of welfare levels. In this respect, the numerical simulations we are now presenting suggest that imperfect information on the consequences of past behavior might not entail a significant welfare cost. This result softens concerns about the use of the *Precautionary Principle* in practice. The two trajectories with and without observability mainly differ at early dates but are very close afterwards; and this result holds under a broad range of scenarios.

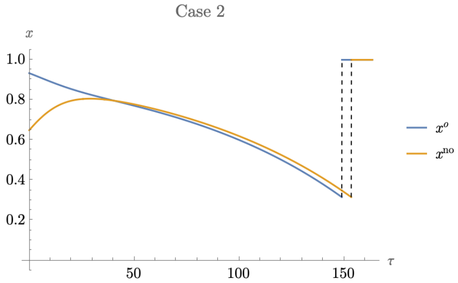
Scenario 0. To fix ideas, suppose that the highest possible value of the tipping point is known to be located at $\overline{X} = 100$ while the myopic action is $x^m = \zeta = 1$. The interest rate is $r = 0.01$. We also assume that before the tipping point the rate of arrival of a catastrophe is very small, namely $\theta_0 = 0.001$, while it jumps to $\theta_1 = 0.1$ afterwards.[24] The tipping point is reached at date $\overline{T}^k = 150.257$, which is significantly higher than in the myopic scenario that, thanks to our normalization, corresponds to $\overline{T}^m = 100$. Next figure represents the action profile $x^k(\tau) = \sigma^e(X^k(\tau), 1)$ where $X^k(\tau) = \int_0^\tau x^k(\tilde{\tau})d\tilde{\tau}$. The intuition is that decreasing the action pushes back the switch to the higher risk scenario, but it comes at a utility cost. Because of discounting, the decision-maker decreases the action over time before reaching the tipping point. Similar patterns are found under uncertainty on the tipping point.



Scenario 1. Suppose that there is an equal probability to pass the tipping point at zero and at $\overline{X}$, i.e., $q = \frac{1}{2}$. Under uncertainty, we expect to find different dates at which the upper bound $\overline{X}$ is now reached depending on whether deviations are observable or not. In fact, we compute $\overline{T}^0 = 149.026$ and, as expected, a higher value $\overline{T}^{no} = 153.535$.

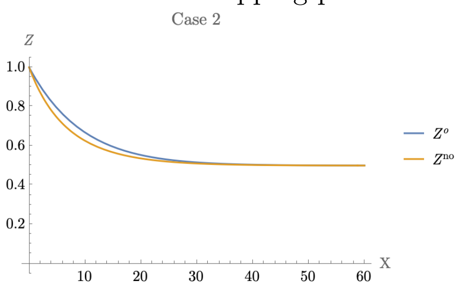
[24] This latter value is actually consistent with those chosen by Besley and Dixit (2019) in a related context, although those authors posit that the arrival rate is a smooth and nonlinear function while we adopt a step function.

Yet, the difference is less than 3 %. This minor difference comes from the fact that the two action profiles $x^o(\tau)$ and $x^{no}(\tau)$ are themselves close to each other. Interestingly, the action path is non-monotonous for the non-observable case. The intuition is that at the beginning, decision-makers are rather pessimistic and decide to enjoy flow payoffs by increasing actions. Conditional on no catastrophe having yet happened, after a while it becomes more likely that the tipping was not in fact at 0, and so that actions are again reduced to push back the switch.

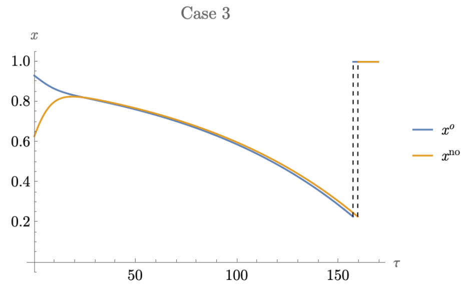


Although quite similar after a while, actions in the two scenarios mostly differ at the start. In the non-observable scenario, decision-makers start with a very low action and then increase actions over a first phase as they become more pessimistic and believe that the tipping point is more likely to have been passed. In a second phase, decision-makers adopt actions which are close to those in the observable scenario. The regime survival ratios in both scenarios become very flat after a while and the existing pessimistic stigma that pertains to the observable-deviation scenario has not enough magnitude to significantly distinguish trajectories in the two scenarios. In other words, for most of the trajectory, there is little impact of observing past deviations on actual choices.

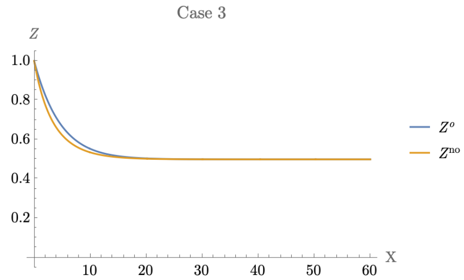
Turning now to the regime survival ratios, we first observe that, since actions are higher when deviations are observable, the stock with observable deviations $X^o(\tau)$ accumulates over time faster than the stock $X^{no}(\tau)$ with non-observable deviations. Let denote by $\tau^{-1,o}(X)$ and $\tau^{-1,no}(X)$ the corresponding inverse functions. Using (4.23) and (6.12) allows us to recover the expressions of the regime survival ratios in terms of X and to check that $Z^{no}(X) = Z(\tau^{-1,no}(X)) \leq Z(\tau^{-1,o}(X)) = Z^o(X)$ as confirmed on next figure. Notice that in this scenario and the following ones, the asymptote is at 0.5 because after enough time DMs are sure that the tipping point was not at 0, but at $\overline{X}$.



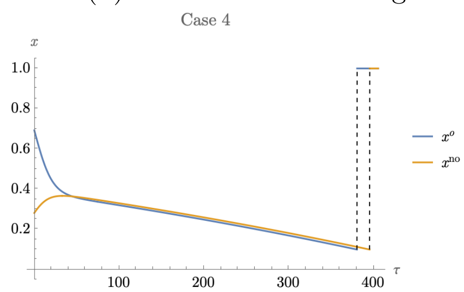
Scenario 2: Increase in the rate of arrival of a catastrophe. Keeping all other parameters as in Scenario 1, consider increasing the rate of arrival of a catastrophe up to $\theta_1 = 0.2$. This change increases delays before reaching the maximal value of the tipping point but does not change the fact that trajectories with observable and non-observable deviations are very close. The switching times now differ by less than 1.5%, at $\overline{T}^0 = 157.147$ and $\overline{T}^{no} = 159.494$ and the action profiles $x^o(\tau)$ and $x^{no}(\tau)$ are described as follows.



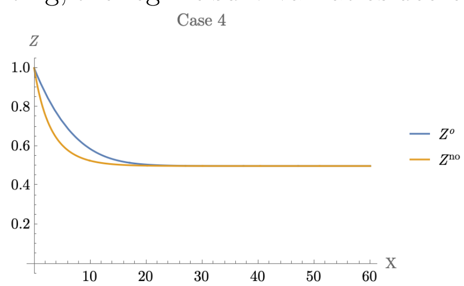
The main effect of increasing the rate of arrival of a catastrophe is to make regime survival ratios decrease faster as shown below.



Scenario 3: Zero discounting. Consider now the case of zero discounting (i.e., $r = 0$) as advocated by Stern (2007) and suppose again that $\theta_1 = 0.1$. In this scenario, the sole source of discounting comes from the probability of a catastrophe that suppresses future payoffs. Because this event is unlikely before having crossed the tipping point, low actions are now chosen in a first phase that lasts longer. Indeed, we find $\overline{T}^0 = 380.429$ and $\overline{T}^{no} = 395.302$. Yet, the difference between the scenarios with and without observable deviations is mild; those delays now differing by less than 4%. With almost no-discounting, the future matters a lot and the *Irreversibility Effect* is quite significant. As a result, both $x^o(\tau)$ and $x^{no}(\tau)$ remain low for a long time while still being close.

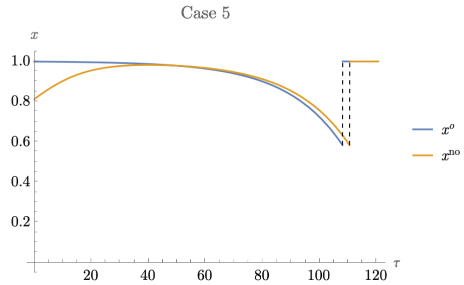


With almost no-discounting, the regime survival ratios decrease slowly as shown below.

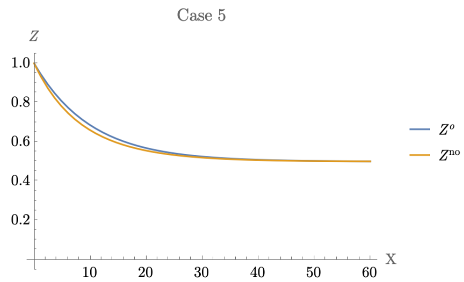


Scenario 4: High discounting. In his critique of the *Stern Review*, Weitzman (2007) advocated a much higher discount rate, namely $r = 0.05$. The first consequence of high discounting is to shorten the delays before reaching $\overline{X}$. We find that the switching times,

namely $\overline{T}^0 = 108.114$ and $\overline{T}^{no} = 110.596$, differ by less than 2.3%. The second consequence is that both actions $x^o(\tau)$ and $x^{no}(\tau)$ although still nearby are now closer to the myopic solution for a long time. Indeed, a high discount rate makes behaving myopically more attractive; leaving distortions needed to satisfy the irreversibility constraint only for the very last periods before reaching $\overline{X}$.



With high discounting, the regime survival ratios in both scenarios are now almost the same.



8. CONCLUDING REMARKS

We have considered a dynamic decision-making problem with irreversibility and uncertainty. Increasing current actions makes it more likely to pass a tipping point and thus increases the likelihood of an environmental catastrophe but the location of such tipping point remains unknown through the process. The optimal trajectory follows a feedback rule that depends not only on the stock of past actions but also on beliefs on whether the tipping point has been passed or not. This trajectory can be implemented as a decentralized equilibrium where decision-makers, acting at different points in time and sharing the same objectives, have limited commitment power and adopt a *Stock-Markov* feedback rule that only depends on stock. This implementation requires that impulse deviations are observable by followers. Indeed, upon observing such deviations, future decision-makers are able to reconstruct the evolution of beliefs and act as what a planner would do at the optimal trajectory. Instead, when impulse deviations are non-observable, the equilibrium feedback rule entails more prudent actions. When actions have been kept low in the past, decision-makers remain quite optimistic on the fact that the tipping point has not been passed yet. In response, they also refrain from taking large actions to avoid any irreversible move.

This framework has allowed us to discuss the relevance of the *Precautionary Principle* that states that one should not act when the consequences of those acts remain unknown. Numerical simulations nevertheless suggest that a trajectory so constrained remains close to the optimum under broad circumstances. The lack of information across decision-makers might thus not be so damageable to society, softening concerns regarding the use of the *Precautionary Principle*.

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APPENDIX A: VALUE FUNCTION AND FEEDBACK RULE

Beliefs

We start by presenting the evolution of the posterior density function $f(\widetilde{X}|t, \mathbf{x}^t)$. For future reference, notice that, as times passes, a stock process $\hat{X}(t;0)$ of the form

$$\hat{X}(t;0) = \int_0^t x(s)ds. \tag{A.1}$$

goes through various possible values $\widetilde{X}$ of the tipping point. We may thus also describe this process by the time $T(\widetilde{X};0)$ at which this stock reaches a level $\widetilde{X}$.[25]

LEMMA A.1 *The posterior density function $f(\widetilde{X}|t, \mathbf{x}^t)$ conditional on not having a catastrophe up to date t following history $\mathbf{x}^t$ satisfies:*

$$f(\widetilde{X}|t, \mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \textit{if } \hat{X}(t;0) \leq \widetilde{X} \\ \frac{e^{-\theta_0 t} e^{-\Delta(t - T(\widetilde{X};0))}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \textit{otherwise.} \end{cases} \tag{A.2}$$

PROOF OF LEMMA A.1: We first compute the probability of survival $H(t, \mathbf{x}^t)$ as (4.1). The first term on the r.-h.s. of (4.1) stems for the probability that the tipping point is below $\hat{X}(t;0)$,

[25] If $\hat{X}(t;0)$ is smooth, increasing and differentiable in t with no flat part, $T(\widetilde{X};0)$ is itself increasing and smooth and differentiable with a finite derivative.

and the rate of survival then jumps up to θ_1 at a date $T(\widetilde{X};0)$ before date t. The second term is the probability that the tipping point is above $\hat{X}(t;0)$ and the rate of arrival of a catastrophe is still θ_0. Denote these terms respectively by P_{1t} and P_{2t}. We immediately compute

$$P_{2t} = (1 - F(\hat{X}(t;0)))e^{-\theta_0 t}. \tag{A.3}$$

Changing variables and letting $\hat{X}(\tau;0) = \widetilde{X}$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0)d\tau = d\widetilde{X}$, we rewrite

$$P_{1t} = \int_0^{\hat{X}(t;0)} f(\widetilde{X})e^{-\theta_0 T(\widetilde{X};0)}e^{-\theta_1(t-T(\widetilde{X};0))}d\widetilde{X} = \int_0^t f(\hat{X}(\tau;0))\frac{\partial \hat{X}}{\partial \tau}(\tau;0)e^{-\theta_0\tau}e^{-\theta_1(t-\tau)}d\tau.$$

Integrating by parts yields

$$P_{1t} = e^{-\theta_0 t}\left(\left[F(\hat{X}(\tau;0))e^{\Delta(\tau-t)}\right]_0^t - \Delta\int_0^t F(\hat{X}(\tau;0))e^{\Delta(\tau-t)}d\tau\right). \tag{A.4}$$

Inserting (A.3) and (A.4) into (4.1) finally yields the expression of the probability of survival up to date t in (4.2). From this expression, we compute the conditional density

$$f(\widetilde{X}|t,\mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t,\mathbf{x}^t)}f(\widetilde{X}) & \text{if } \hat{X}(t;0) \le \widetilde{X} \\ \frac{e^{-\theta_0 T(\widetilde{X};0)}e^{-\theta_1(t-T(\widetilde{X};0))}}{H(t,\mathbf{x}^t)}f(\widetilde{X}) & \text{otherwise.} \end{cases}$$

Simplifying yields (A.2). *Q.E.D.*

Value Function

PROOFS OF LEMMA 1: Following history $\mathbf{x}^t$, the stock $\hat{X}(\tau;X,t)$ evolves as

$$\hat{X}(\tau;X,t) = X + \int_t^\tau x(s)ds. \tag{A.5}$$

with a stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau\ge t}$. Let $T(\widetilde{X};X,t)$ accordingly denote the inverse function defined for $\widetilde{X} \ge X$. The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ can be written as

$$\begin{aligned} \hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,X(\cdot)\text{ s.t. (A.5)}} &\int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-t)}u(x(\tau))d\tau\right)f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X} \\ &+\int_X^{+\infty}\left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)}e^{-\theta_0(\tau-t)}u(x(\tau))d\tau\right. \\ &\left.+e^{-\theta_0(T(\widetilde{X};X,t)-t)}\int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-T(\widetilde{X};X,t))}u(x(\tau))d\tau\right)f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}. \end{aligned} \tag{A.6}$$

Taking into account the expression of the conditional density given in (A.2), we rewrite the expression of $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ in (A.6) as

(A.7)

$$\begin{aligned} e^{\theta_0 t}H(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,X(\cdot)\text{ s.t. (A.5)}} &\int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-t)}u(x(\tau))d\tau\right)e^{-\Delta(t-T(\widetilde{X};0))}f(\widetilde{X})d\widetilde{X} \\ &+\int_X^{+\infty}\left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)}e^{-\theta_0(\tau-t)}u(x(\tau))d\tau\right. \\ &\left.+e^{-\theta_0(T(\widetilde{X};X,t)-t)}\int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)}e^{-\theta_1(\tau-T(\widetilde{X};X,t))}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X}. \end{aligned}$$

Let

$$\mathcal{I}_1 = \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X}$$

which rewrites as

$$\mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} \right). \tag{A.8}$$

Changing variables and letting $\hat{X}(\tau;0) = \widetilde{X}$ for $\tau \leq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0)d\tau = d\widetilde{X}$, we also rewrite

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} = \int_0^t e^{-\Delta(t-\tau)} f(\hat{X}(\tau;0)) \frac{\partial \hat{X}}{\partial \tau}(\tau;0) d\tau.$$

Integrating by parts, yields

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} = e^{-\Delta t} \left(\left[F(\hat{X}(\tau;0)) e^{\Delta\tau} \right]_0^t - \Delta \int_0^t F(\hat{X}(\tau;0)) e^{\Delta\tau} d\tau \right)$$

$$= F(X) - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau;0)) e^{\Delta\tau} d\tau$$

where the last equality follows from $\hat{X}(t;0) = X$. Inserting into (A.8) yields

$$\mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(F(X) - \Delta e^{-\Delta t} \int_0^t F(X(s;0)) e^{\Delta s} ds \right). \tag{A.9}$$

We now compute

$$\mathcal{I}_2 = \int_X^{+\infty} \left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$

$$\left. + e^{-\theta_0(T(\widetilde{X};X,t)-t)} \int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-T(\widetilde{X};X,t))} u(x(\tau)) d\tau \right) f(\widetilde{X}) d\widetilde{X}.$$

Changing variables and letting $\hat{X}(\tau;X,t) = \widetilde{X}$ for $\tau \geq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;X,t)d\tau = d\widetilde{X}$ and $\hat{X}(t;X,t) = X$, we also rewrite

$$\mathcal{I}_2 = \int_t^{+\infty} \left(\int_t^{\tau} e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) f(\hat{X}(\tau;X,t)) \frac{\partial \hat{X}}{\partial \tau}(\tau;X,t) d\tau.$$

Integrating by parts yields

$$\mathcal{I}_2 = \left[F(\hat{X}(\tau;X,t)) \left(\int_t^{\tau} e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty} \tag{A.10}$$

$$-\Delta \int_t^{+\infty} F(\hat{X}(\tau;X,t)) e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \bigg) d\tau.$$

Using that $\lim_{\tau \to +\infty} F(\hat{X}(\tau;X,t)) = 1$ if $\lim_{\tau \to +\infty} \hat{X}(\tau;X,t) = +\infty$ (which holds when the minimal action is positive at any point of time as we will see below), we get

$$\mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \tag{A.11}$$

$$-\Delta \int_t^{+\infty} F(\hat{X}(\tau; X, t)) e^{\Delta(\tau - t)} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau.$$

Integrating by parts, we obtain

$$\int_t^{+\infty} F(\hat{X}(\tau; X, t)) e^{\Delta \tau} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau$$

$$= \left[\left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty}$$

$$+ \int_\tau^{+\infty} e^{-\lambda_1(\tau - t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau$$

$$= \int_\tau^{+\infty} e^{-\lambda_1(\tau - t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Inserting into (A.11), we thus obtain

$$\text{(A.12)} \quad \mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds$$

$$-\Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau - t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Summing up (A.9) and (A.12) and taking into account that $\hat{X}(s; X, t)$ for $s \geq t$ is the continuation of the trajectory $\hat{X}(s; 0)$, i.e., $\hat{X}(s; X, t) \equiv \hat{X}(s; 0, 0) = \hat{X}(s; 0)$ (where the last equality slightly abuses notation) for $s \geq t$, yields

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau - t)} u(x(\tau)) d\tau - \Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau - t)} \left(\int_0^\tau F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau)) d\tau$$

and thus

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau - t)} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Changing variables and setting $\tau' = \tau - t$ yields

$$\text{(A.13)} \quad \mathcal{I} = \int_0^{+\infty} e^{-\lambda_0 \tau'} \left(1 - \Delta e^{-\Delta(\tau' + t)} \int_0^{\tau' + t} F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau' + t)) d\tau'.$$

Generalizing (4.2) to paths that go till date $t + \tau$, we observe that the probability of survival up to date $t + \tau$ can be expressed in terms of the action plan $\mathbf{x}^{t+\tau}$ followed up to that date (that plan includes all past actions taken up to date t, namely $\mathbf{x}^t$, and the actions planned from date t on $\mathbf{x}_t^{t+\tau}$) as

$$\text{(A.14)} \quad H(t + \tau, \mathbf{x}^{t+\tau}) = e^{-\theta_0(t+\tau)} \left(1 - \Delta e^{-\Delta(t+\tau)} \int_0^{t+\tau} F(\hat{X}(s; 0)) e^{\Delta s} ds \right).$$

Inserting into (A.13) and changing the name of dummy variables yields

$$\text{(A.15)} \quad \mathcal{I} = e^{\theta_0 t} \int_0^{+\infty} e^{-r\tau} H(t + \tau, \mathbf{x}^{t+\tau}) u(x(\tau + t)) d\tau.$$

Inserting into (A.7) yields

$$e^{\theta_0 t}H(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,\hat{X}(\cdot)} \int_0^{+\infty} e^{-\lambda_0\tau}e^{\theta_0(t+\tau)}H(t+\tau,\mathbf{x}^{t+\tau})u(x(t+\tau))d\tau$$

$$\text{s.t. } \hat{X}(t+\tau;0) = X + \int_0^{\tau} x(t+s)ds \text{ and } X = \int_0^{\tau} \overline{x}(s)ds.$$

which can be written as

$$\hat{Z}(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,\hat{X}(\tau;X,t)=X+\int_t^{\tau} x(s)ds} \int_0^{+\infty} e^{-\lambda_0\tau}\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})u(x(t+\tau))d\tau. \quad \text{(A.16)}$$

and, finally, (4.4) with the definition of $\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})$ in (4.3). *Q.E.D.*

Next proposition provides some properties of the value function $\mathcal{V}^e(X,Z)$. At a higher stock, $\mathcal{V}^e(X,Z)$ is necessarily lower since the irreversibility constraints become more stringent as X comes closer to $\overline{X}$.

Proposition A.1 *There exists a solution to the optimization problem (4.11). $Z\mathcal{V}^e(X,Z)$ is non-increasing in X, convex in Z, Lipschitz-continuous and thus a.e. differentiable.*

Proof of Proposition A.1: We first define $\mathcal{W}^e(X,Z)$ as

$$\mathcal{W}^e(X,Z) = Z\mathcal{V}^e(X,Z).$$

Inserting (4.7) into the r.-h.s. of (4.11), we thus rewrite

$$\mathcal{W}^e(X,Z) = \max_{\mathbf{x},X(\cdot),T \text{ s.t. } (4.5), X(0)=X,\ X(T)=\overline{X}} (Z-1)\left(\int_0^T e^{-\lambda_0\tau}e^{-\Delta\tau}u(x(\tau))d\tau\right. \quad \text{(A.17)}$$

$$\left.+\lambda_1\mathcal{V}_\infty\int_T^{\infty} e^{-\lambda_0\tau}e^{-\Delta\tau}d\tau\right) + \int_0^T e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^{\tau} F(X(s))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

$$+\int_T^{+\infty} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^{\tau} F(X(s))e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau.$$

Existence. Existence of a solution to the optimization problem (A.17) follows from applying Filipov-Cesari Theorem with free final time (see Seierstad and Sydsaeter, 1987, Theorem 12, p. 145). To check that all conditions for this theorem are satisfied, first observe that $\mathcal{X}$ is closed and bounded, while X is bounded above by $\overline{X}$ and Z is also bounded ($Z \in [0,1]$). Denote

$$N(X,Z,\mathcal{X},\tau) = \{e^{-\lambda_0\tau}Zu(x)+\gamma \le 0, x, \Delta(1-F(X)-Z); \gamma \le 0, x \in \mathcal{X}\}.$$

Let us check that $N(X,Z,\mathcal{X},\tau)$ is convex for each (X,Z,τ). Take a pair $(x_1,x_2) \in N(X,Z,\mathcal{X},\tau)\times N(X,Z,\mathcal{X},\tau)$, i.e., there exist $\gamma_i \le 0$ such that $e^{-\lambda_0\tau}Zu(x_i)+\gamma_i \le 0$. Consider now $\lambda x_1+(1-\lambda)x_2$ for $\lambda \in [0,1]$ and observe that

$$e^{-\lambda_0\tau}Zu(\lambda x_1+(1-\lambda)x_2) \le e^{-\lambda_0\tau}Z(u(\lambda x_1+(1-\lambda)x_2)-\lambda u(x_1)-(1-\lambda)u(x_2))-\lambda\gamma_1-(1-\lambda)\gamma_2.$$

Define $\gamma = \lambda\gamma_1+(1-\lambda)\gamma_2+e^{-\lambda_0\tau}Z(\lambda u(x_1)+(1-\lambda)u(x_2)-u(\lambda x_1+(1-\lambda)x_2))$ and observe that $\gamma \le 0$ since u is concave and $\gamma_i \le 0$. Moreover, we have

$$e^{-\lambda_0\tau}Zu(\lambda x_1+(1-\lambda)x_2)+\gamma \le 0.$$

Hence, $N(X,Z,\mathcal{X},\tau)$ is convex as requested. From Filipov-Cesari Theorem, an optimal arc thus exists. Let denote by $(X^e(\tau;X,Z), Z^e(\tau;X,Z), x^e(\tau;X,Z), \overline{T}^e(\tau;X,Z))$ such an arc.

Properties. Fixing an action path $\mathbf{x}$ and taking $X' \geq X$, the corresponding stocks satisfy $X(s;X) \leq X(s;X')$. The r.-h.s. of (A.17) is thus lower at X' for any action path. Taking the max-operator proves that $\mathcal{W}^e(X,Z)$ is non-increasing in X. From (A.17), it also follows that $\mathcal{W}^e(X,Z)$ is convex as a maximum of linear functions of Z.

Consider an alternative pair (X',Z'). Because an arc which is optimal for (X',Z'), say $(X^e(\tau;X',Z'), Z^e(\tau;X',Z'), x^e(\tau;X',Z'), \overline{T}^e(X',Z'))$, is weakly suboptimal for (X,Z), the following inequality holds:

$$\mathcal{W}^e(X,Z) \geq (Z-1)\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}e^{-\Delta\tau}u(x^e(\tau;X',Z')d\tau + \lambda_1\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^{\infty} e^{-\lambda_0\tau}e^{-\Delta\tau}d\tau\right)$$
$$+\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)u(x^e(\tau;X',Z'))d\tau$$
$$+\int_{\overline{T}^e(X',Z')}^{+\infty} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau.$$

We express the r.-h.s. in terms of $\mathcal{W}^e(X',Z')$ to get:

(A.18)
$$\mathcal{W}^e(X,Z)-\mathcal{W}^e(X',Z') \geq (Z-Z')\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_1\tau}u(x^e(\tau;X',Z'))d\tau + \lambda_1\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^{\infty} e^{-\lambda_1\tau}d\tau\right)+$$
$$\Delta\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}\left(\int_0^\tau\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)\right.\right.\right.$$
$$\left.\left.\left.-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)u(x^e(\tau;X',Z'))d\tau\right)$$
$$+\Delta\left(\int_{\overline{T}^e(X',Z')}^{\infty} e^{-\lambda_0\tau}\left(\int_0^\tau\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)\right.\right.\right.$$
$$\left.\left.\left.-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau\right).$$

Permuting the roles of (X,Z) and (X',Z'), we deduce a similar inequality. Putting together those conditions implies

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X,Z)| \leq \mathcal{V}_\infty(\|f\|_\infty|X'-X|+|Z'-Z|).$$

From which, we deduce that there exists $k = 2\mathcal{V}_\infty\max\{\|f\|_\infty, 1\}$ such that

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X,Z)| \leq k||(X',Z')-(X,Z)||)$$

where $||\cdot||$ is the Euclidian norm. $\mathcal{W}^e(X,Z)$ is Lipschitz continuous and thus a.e. differentiable.
Q.E.D.

For future reference, we now define DM's payoff along an optimal arc $(X^e(\tau;X,Z), Z^e(\tau;X,Z))$ for the stock and the regime survival ratio starting from arbitrary initial conditions (X,Z) in case the regime switch has already occurred as

$$\text{(A.19)} \qquad \varphi^e(X,Z) = \int_0^{\overline{T}^e(X,Z)} e^{-\lambda_1\tau}u(\sigma^e(X^e(\tau;X,Z), Z^e(\tau;X,Z)))d\tau + e^{-\lambda_1\overline{T}^e(X,Z)}\mathcal{V}_\infty$$

where $\overline{T}^e(X,Z)$ is the date at which the highest possible value of the tipping point is reached, namely $X^e(\overline{T}^e(X,Z);X,Z) = \overline{X}$. Payoffs are discounted at a rate λ_1 once the tipping point has been passed. When $X \geq \overline{X}$, DM knows for sure that it has been the case and adopts the myopic action with payoff $\mathcal{V}_\infty$. Beliefs then evolve according to (4.8). Because $\varphi^e(X,Z)$ is computed when discounting payoffs at rate λ_1, while $\mathcal{V}^e(X)$ is computed when discounting at a lower rate λ_0 over a first phase, we necessarily have $\mathcal{V}^e(X,Z) \geq \varphi^e(X,Z)$. Although DM ignores having passed the tipping point, he knows that, if that happened, continuation payoffs are lower.

Proof of Proposition 1: Characterization. We start by characterizating $\mathcal{W}^e(X,Z)$ by means of an Hamilton-Bellman-Jacobi equation.

Proposition A.2 *At any point of differentiability, $\mathcal{W}^e(X,Z)$ that solves (A.17) satisfies the following Hamilton-Bellman-Jacobi partial differential equation:*

(A.20)
$$\lambda_0\mathcal{W}^e(X,Z) = \lambda_1\mathcal{V}_\infty Z + \zeta\frac{\partial\mathcal{W}^e}{\partial X}(X,Z) + \frac{1}{2Z}\left(\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)\right)^2 + \Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z).$$

The feedback rule is given by

$$\sigma^e(X,Z) = \zeta + \frac{1}{Z}\frac{\partial\mathcal{W}^e}{\partial X}(X,Z). \tag{A.21}$$

Moreover, we have

$$\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z) = \varphi^e(X,Z). \tag{A.22}$$

Proof of Proposition A.2: For the sake of completeness and for future references, we remind below the well-known derivation of the Hamilton-Bellman-Jacobi equation satisfied by $\mathcal{W}^e(X,Z)$. Consider $Z \in [0,1]$. Using the *Dynamic Programming Principle*, $\mathcal{W}^e(X,Z)$ satisfies

$$\mathcal{W}^e(X,Z) = \sup_{\mathcal{A}}\int_0^\varepsilon e^{-\lambda_0 t}Z(t)u(x(t))dt + e^{-\lambda_0\varepsilon}\mathcal{W}^e(X(\varepsilon;X,Z),Z(\varepsilon;X,Z)). \tag{A.23}$$

Consider now ε small enough and denote by x a fixed action over the interval $[0,\varepsilon]$. From (4.6) and (4.5), we get

$$X(\varepsilon;X,Z) = X + \varepsilon x + o(\varepsilon), Z(\varepsilon;X,Z) = Z + \varepsilon\Delta(1-F(X)-Z) + o(\varepsilon)$$

where $\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0$.

When $\mathcal{W}^e(X,Z)$ is continuously differentiable, we can take a first-order Taylor expansion in ε of the maximand in (A.23) to write it as

$$\mathcal{W}^e(X,Z)+\varepsilon\left(Zu(x) + x\frac{\partial\mathcal{W}^e}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z) - \lambda_0\mathcal{W}^e(X,Z)\right)+o(\varepsilon).$$

Inserting into (A.23) yields the following Hamilton-Bellman-Jacobi equation:

$$\lambda_0\mathcal{W}^e(X,Z) = \sup_{x\in\mathcal{X}}\left\{Zu(x) + x\frac{\partial\mathcal{W}^e}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)\right\}. \tag{A.24}$$

Feedback Rule. The maximand on the r.-h.s. of (A.24) is strictly concave. It immediately follows that the feedback rule $\sigma^e(X,Z)$ is given by (A.21) when interior. Simplifying (A.24) by using the feedback rule (A.21) finally yields (A.20).

Partial Differential Equation. Rewriting the optimality conditions in terms of $\mathcal{V}^e(X,Z)$, (A.20) becomes

$$\lambda_0\mathcal{V}^e(X,Z) = \lambda_1\mathcal{V}_\infty + \zeta\frac{\partial\mathcal{V}^e}{\partial X}(X,Z) + \frac{1}{2}\left(\frac{\partial\mathcal{V}^e}{\partial X}(X,Z)\right)^2 + \frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z).$$

Solving this second-degree equation and keeping the solution that gives a positive feedback rule yields

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,Z) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^e(X,Z) - 2\frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial \mathcal{W}^e}{\partial Z}(X,Z)}. \tag{A.25}$$

Denote the optimal solution to (A.17) by $\left(x^e(\tau;X,Z), X^e(\tau;X,Z), Z^e(\tau;X,Z), \overline{T}^e(X,Z)\right)$. From (A.17), we can write

(A.26)

$$\mathcal{W}^e(X,Z) = \int_0^{\overline{T}^e(X,Z)} e^{-\lambda_0\tau} Z^e(\tau;X,Z) u(x^e(\tau;X,Z))d\tau + Z^e(\overline{T}^e(X,Z);X,Z)e^{-\lambda_0\overline{T}^e(X,Z)}\mathcal{V}_\infty.$$

Integrating (4.6), we obtain

$$\widetilde{Z}^e(\tau;X,Z) = (Z-1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau}\int_0^\tau F(X^e(s;X,Z))e^{\Delta s}ds \quad \forall \tau \geq 0, X, Z \geq 0 \tag{A.27}$$

Applying the Envelope Theorem to (A.17) thus yields

$$\frac{\partial \mathcal{W}^e}{\partial Z}(X,Z) = \varphi^e(X,Z) \tag{A.28}$$

or

$$Z\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z) + \mathcal{V}^e(X,Z) = \varphi^e(X,Z)$$

where $\varphi^e(X,Z)$ is defined as in (A.19). Inserting into (A.25) and simplifying yields

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,Z) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^e(X,Z) - 2\frac{\Delta(1-F(X)-Z)}{Z}\varphi^e(X,Z)}$$

which can be written as (4.12).

Q.E.D.

Q.E.D.

Bounds. For future references, it is useful to provide simple bounds on $\mathcal{V}^e(X,Z)$.

Proposition A.3

$$Z\mathcal{V}_\infty \leq Z\mathcal{V}^e(X,Z) \leq \left(F(X) + (1-F(X))\frac{\lambda_1}{\lambda_0}\right)\mathcal{V}_\infty \quad \forall X \geq 0, \forall Z \in (0,1]. \tag{A.29}$$

Proof of Proposition A.3: Observe that (4.6) and $F(X) \leq F(X^e(\tau;X,Z))) \leq 1$ imply

$$0 \leq \frac{d}{d\tau}\left(Z^e(\tau;X,Z)e^{\Delta\tau}\right) \leq \Delta(1-F(X))e^{\Delta\tau}.$$

Integrating between 0 and τ yields

$$0 \leq Ze^{-\Delta\tau} \leq Z^e(\tau;X,Z) \leq Ze^{-\Delta\tau} + (1-F(X))\left(1-e^{-\Delta\tau}\right).$$

From this and the fact that $0 \leq Z \leq 1$, it follows that

$$0 \leq Ze^{-\Delta\tau} \leq Z^e(\tau;X,Z) \leq F(X)e^{-\Delta\tau} + 1 - F(X) \leq 1. \tag{A.30}$$

Henceforth, the whole trajectory $Z^e(\tau;X,Z)$ always remains in the stable domain $[0,1]$.

From the third inequality in (A.30), taking maximum on the r.-h.s. of (A.17), the r.-h.s. inequality of (A.29) follows. From the first inequality in (A.30), we immediately get the l.-h.s. inequality of (A.29). *Q.E.D.*

Optimal Path

The intertemporal date 0-payoff $\mathcal{V}^e(0,1)$ is achieved by adopting the action profile $\sigma^e(X^e(\tau;0,1)$ for all $\tau \geq 0$ starting from the initial conditions $X = 0$ and $Z = 1$. Next Proposition provides necessary conditions for an optimal arc.

PROPOSITION A.4 *An optimal action path $x^e(t)$ satisfies the following necessary condition:*[26]

$$x^e(\tau) = \zeta - \frac{\Delta e^{\lambda_0 \tau}}{Z^e(\tau)} \int_\tau^{\overline{T}^e} f(X^e(s)) e^{\Delta s} \left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'} u(x^e(s'))ds' \right) ds \tag{A.31}$$

where, along the optimal trajectory, the survival ratio writes as

$$Z^e(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(X^e(\tau)) e^{\Delta \tau} d\tau.$$

$\overline{X}$ is reached at a date $\overline{T}^e < \overline{T}^m$ with

$$\overline{X} = \zeta \overline{T}^e - \int_0^{\overline{T}^e} \frac{\Delta e^{\lambda_0 \tau}}{Z^e(\tau)} \left(\int_\tau^{\overline{T}^e} f(X^e(s)) e^{\Delta s} \left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'} u(x^e(s'))ds' \right) ds \right) d\tau. \tag{A.32}$$

PROOF OF PROPOSITION A.4 : From (4.4), DM's intertemporal payoff writes as

$$\mathcal{V}^e(0,1) \equiv \sup_{\mathcal{A}} \int_0^T e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau + \int_T^{+\infty} e^{-\lambda_0 \tau} Z(\tau) \lambda_1 \mathcal{V}_\infty d\tau. \tag{A.33}$$

EXISTENCE. It immediately follows that there exists a solution to problem (A.33) from the argument for existence in the Proof of Proposition 1.
MAXIMUM PRINCIPLE. Observe that, for $\tau \geq T$, (4.6) implies

$$Z(\tau) = Z(T) e^{-\Delta(\tau - T)} \tag{A.34}$$

and thus the scrap value on the r.-h.s. of the maximand in (A.33) writes as

$$\int_T^{+\infty} e^{-\lambda_0 \tau} Z(\tau) \lambda_1 \mathcal{V}_\infty d\tau = Z(T) e^{-\lambda_0 T} \mathcal{V}_\infty. \tag{A.35}$$

We now define the Hamiltonian for this optimization problem as

$$\mathcal{H}^e(X, Z, x, \tau, \mu, \nu) = e^{-\lambda_0 \tau} Z u(x) + \mu x + \nu \Delta(1 - F(X) - Z) \tag{A.36}$$

where μ and ν are respectively the costate variables for (A.1) and (4.6). The *Maximum Principle* with free final time and scrap value now gives us the following necessary conditions for optimality of an arc $(X^e(\tau), Z^e(\tau), x^e(\tau), \overline{T}^e)$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)
Costate variables. $\mu(\tau)$ and $\nu(\tau)$ are both continuously differentiable on $\mathbb{R}_+$ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial X}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

or

$$\dot{\mu}(\tau) = \Delta f(X^e(\tau)) \nu(\tau) \quad \forall \tau \in \left[0, \overline{T}^e\right]; \tag{A.37}$$

and

$$-\dot{\nu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial Z}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

[26] We slightly abuse notations and omit the dependence on the initial conditions $(0,1)$.

or

$$\dot{\nu}(\tau) = -e^{-\lambda_0 \tau} u(x^e(\tau)) + \Delta\nu(\tau) \quad \forall \tau \in \left[0, \overline{T}^e\right]. \tag{A.38}$$

Transversality conditions. The boundary conditions $X^e(0) = 0$, $X^e(\overline{T}^e) = \overline{X}$ and $Z^e(0) = 1$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \overline{T}^e$ and on $\nu(\tau)$ at $\tau = 0$ only while

$$\nu(\overline{T}^e) = 0. \tag{A.39}$$

Free-end point conditions. The optimality condition with respect to $\overline{T}$ writes as

$$\mathcal{H}^e(X^e(\overline{T}^e), Z^e(\overline{T}^e), x^e(\overline{T}^e), \overline{T}^e, \mu(\overline{T}^e), \nu(\overline{T}^e)) + \frac{d}{dT}\left(Z(T)e^{-\lambda_0 T}\right)_{T=\overline{T}^e} \mathcal{V}_\infty = 0. \tag{A.40}$$

Using (A.36), (A.39), (4.6) taken for $\overline{T}^e$ (with the fact that F has no mass point at $\overline{X}$), namely

$$\dot{Z}(\overline{T}^e) = -\Delta Z(\overline{T}^e), \tag{A.41}$$

Condition (A.40) rewrites as

$$e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e)\left(u(x^e(\overline{T}^{e-})) - \lambda_1 \mathcal{V}_\infty\right) + \mu(\overline{T}^e) x^e(\overline{T}^{e-}) = 0 \tag{A.42}$$

or

$$-\frac{1}{2} e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e)(x^e(\overline{T}^{e-}) - \zeta)^2 + \mu(\overline{T}^e) x^e(\overline{T}^{e-}) = 0 \tag{A.43}$$

where $x^e(\overline{T}^{e-})$ denotes the l.-h. side limit of $x^e(\tau)$ as $\tau \to \overline{T}^{e-}$.
Control variable $x^e(\tau)$.

$$x^e(\tau) \in \arg\max_{x \geq 0} \mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \mu(\tau), \nu(\tau)).$$

Because $\mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \tau, \mu(\tau), \nu(\tau))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}^e}{\partial x}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau)) = 0$$

or

$$x^e(\tau) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau)}{Z^e(\tau)}. \tag{A.44}$$

Characterization. Inserting (A.44) taken for $\overline{T}^e$ into (A.43) yields

$$\frac{e^{\lambda_0 \overline{T}^e} \mu^2(\overline{T}^e)}{2Z^e(\overline{T}^e)} + \mu(\overline{T}^e)\zeta = 0.$$

The only solution consistent with a non-negative action at date $\overline{T}^e$ is thus

$$\mu(\overline{T}^e) = 0. \tag{A.45}$$

From there, it follows that the optimal action is continuous at $\overline{T}^e$, namely

$$x^e(\overline{T}^{e-}) = x^e(\overline{T}^{e+}) = \zeta. \tag{A.46}$$

The solution for (A.38) that satisfies the transversality condition (A.39) is

$$\nu(\tau) = e^{\Delta \tau} \int_\tau^{\overline{T}^e} e^{-\lambda_1 s} u(x^e(s)) ds. \tag{A.47}$$

Inserting into (A.37) and integrating yields

$$\mu(\tau) = \mu(\overline{T}^e) - \int_\tau^{\overline{T}^e} \Delta f(X^e(s))e^{\Delta s}\left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'}u(x^e(s'))ds'\right)ds$$

or, using (A.45),

$$\mu(\tau) = -\int_\tau^{\overline{T}^e} \Delta f(X^e(s))e^{\Delta s}\left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'}u(x^e(s'))ds'\right)ds. \tag{A.48}$$

Inserting into (A.44), we obtain (A.31). Finally, the value of $\overline{T}^e$ is obtained when $\int_0^{\overline{T}^e} x^e(\tau)d\tau = \overline{X}$ or (A.32). That $\overline{T}^e < \overline{T}^m$ is immediate.

Q.E.D.

Proofs of Proposition 2 : The Hamilton-Bellman-Jacobi equation (4.16) and the optimal feedback rule (4.18). Iimmediately follows from Proposition 1 taken at $Z = 1$.
Comparative Statics. From (4.16), we have $\frac{\partial \mathcal{V}^e}{\partial X}(X,1) \leq 0$ if and only if $\mathcal{V}^e(X,1) \leq \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ Observe that $\mathcal{V}^e(\overline{X},1) < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ because of (4.17). Moreover, $\mathcal{V}^e(X,1)$ were to cross $\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ at $X_1 < \overline{X}$, we would have $\frac{\partial \mathcal{V}^e}{\partial X}(X_1,1) = 0$. Observe that $\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ is a constant solution to (4.16). Suppose that $\mathcal{V}^e(X,1)$ were to cross $\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ at $X_1 < \overline{X}$. By Cauchy-Lipschitz Theorem, the only solution to (4.16) which is such $\mathcal{V}^e(X_1,1) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ is such that $\mathcal{V}^e(X,1) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ for all $X \in [0,\overline{X}]$. This would contradict the boundary condition (4.17). Hence, necessarily, $\mathcal{V}^e(X,1) < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ for all X. From (4.16), $\frac{\partial \mathcal{V}^e}{\partial X}(X,1) < 0$ for $X < \overline{X}$. From (4.17), we thus have $\mathcal{V}^e(X,1) > \mathcal{V}_\infty$) for $X < \overline{X}$.

Turning now to the optimal action. The r.-h.s. inequality of (4.21) follows from (4.18) and $\frac{\partial \mathcal{V}^e}{\partial X}(X,1) < 0$ for $X < \overline{X}$. The l.-h.s. inequality follows from the l.-h.s. inequality in (**??**), together with (4.16) and (4.18).

Differentiating (4.16) with respect to X yields

$$\left(1 + \frac{\zeta}{\frac{\partial \mathcal{V}^e}{\partial X}(X,1)}\right)\frac{\partial^2 \mathcal{V}^e}{\partial X^2}(X,1) = \lambda_0. \tag{A.49}$$

Because $\frac{\partial \mathcal{V}^e}{\partial X}(X,1) < 0$ for $X \in [0,\overline{X})$ and $\sigma^e(X,1) = \frac{\partial \mathcal{V}^e}{\partial X}(X,1) + \zeta > 0$, we deduce that $\frac{\partial^2 \mathcal{V}^e}{\partial X^2}(X,1) < 0$ for $X \in [0,\overline{X})$ and thus $\sigma^e(X,1)$ is decreasing. *Q.E.D.*

APPENDIX B: *SME* WITH OBSERVABLE IMPULSE DEVIATIONS

For further reference, we now state some preliminary Lemmas.

Lemma B.1

$$\frac{\partial X^o}{\partial X}(\tau;X) = \frac{\sigma^o(X^o(\tau;X))}{\sigma^o(X)} = \frac{\frac{\partial X^o}{\partial \tau}(\tau;X)}{\sigma^o(X)}. \tag{B.1}$$

Lemma B.2

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = \sigma^o(X^o(\tau;X))\left(\frac{x}{\sigma^o(X)} - 1\right). \tag{B.2}$$

Lemma B.3 *$Z(\tau;X)$ and $Z^o(X)$ satisfy the following conditions*

$$\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) = \frac{\partial Z}{\partial \tau}(\tau;X) \quad \forall \tau \geq 0, X \geq 0, \tag{B.3}$$

$$\sigma^o(X)\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \text{ with } Z^o(0) = 1. \tag{B.4}$$

$Z^o(X) \geq 1 - F(X)$ for all X with equality at $X = 0$ only, and thus $\dot{Z}^o(X) \leq 0$ when $\sigma^o(X) > 0$.

Next Lemma provides a characterization of any continuously differentiable *SME* with *Stock-Markov* value function and feedback rule $(\mathcal{V}^o(X), \sigma^o(X)))$.

LEMMA B.4 *If $\mathcal{V}^o(X)$ is continuously differentiable, the following necessary conditions hold:*

$$0 = \max_{x\in\mathcal{X}} \frac{\partial\hat{\mathcal{V}}}{\partial\varepsilon}(x, 0, X), \tag{B.5}$$

$$\sigma^o(X) \in \arg\max_{x\in\mathcal{X}} \frac{\partial\hat{\mathcal{V}}}{\partial\varepsilon}(x, 0, X). \tag{B.6}$$

We are now ready to characterize the *Stock-Markov* value function.

PROOF OF PROPOSITION 4: We define

$$\mathcal{W}^o(X) = Z^o(X)\mathcal{V}^o(X) \tag{B.7}$$

where

$$\mathcal{W}^o(X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X)))d\tau. \tag{B.8}$$

Next lemma turns to the properties of $\mathcal{V}^o(X)$ and $\varphi^o(X)$.

LEMMA B.5 *$\mathcal{V}^o(X)$ and $\varphi^o(X)$ satisfy the following system of first-order differential equations:*

$$\sigma^o(X)\left(\dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X)\right) = \lambda_0\mathcal{V}^o(X) - u(\sigma^o(X)), \tag{B.9}$$

$$\sigma^o(X)\dot{\varphi}^o(X) = \lambda_1\varphi^o(X) - u(\sigma^o(X)). \tag{B.10}$$

PROOF OF LEMMA B.5: Differentiating (B.8) with respect to X yields

$$\dot{\mathcal{W}}^o(X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau; X) u'(\sigma^o(X^o(\tau; X)))\dot{\sigma}^o(X^o(\tau; X))\frac{\partial X^o}{\partial X}(\tau; X)d\tau$$
$$+ \int_0^{+\infty} e^{-\lambda_0\tau} \frac{\partial Z}{\partial X}(\tau; X) u(\sigma^o(X^o(\tau; X)))d\tau.$$

Using (B.1), we rewrite this condition as

$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau; X) u'(\sigma^o(X^o(\tau; X)))\dot{\sigma}^o(X^o(\tau; X))\frac{\partial X^o}{\partial \tau}(\tau; X)d\tau \tag{B.11}$$
$$+ \int_0^{+\infty} e^{-\lambda_0\tau} \sigma^o(X)\frac{\partial Z}{\partial X}(\tau; X) u(\sigma^o(X^o(\tau; X)))d\tau.$$

Integrating by parts the first integral above, we find

(B.12)
$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \left[e^{-\lambda_0\tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X)))\right]_0^{+\infty} + \lambda_0 \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X)))d\tau$$
$$+ \int_0^{+\infty} e^{-\lambda_0\tau} \left(\sigma^o(X)\frac{\partial Z}{\partial X}(\tau; X) - \frac{\partial Z}{\partial \tau}(\tau; X)\right) u(\sigma^o(X^o(\tau; X)))d\tau.$$

Using (B.3) and simplifying yields

$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \lambda_0\mathcal{W}^o(X) - Z^o(X)u(\sigma^o(X)) \quad \forall X. \tag{B.13}$$

Using the definition of $\mathcal{W}^o(X)$ in (B.7) and simplifying yields (B.9).

Using (5.7) and differentiating with respect to X yields

$$\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial X}(\tau;X)d\tau.$$

Using (B.1), we rewrite this condition as

$$\sigma^o(X)\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau. \tag{B.14}$$

Integrating by parts we obtain

$$\int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau = \left[e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty}$$

$$+\lambda_1\int_0^{+\infty} e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))d\tau = -u(\sigma^o(X)) + \lambda_1\varphi^o(X).$$

Inserting into (B.14) ends the proof. *Q.E.D.*

By adopting the deviation (5.8)-(5.9), the regime survival ratio would also change as (5.10). We can thus write the benefit of a deviation as

$$\mathcal{W}(\varepsilon, x; X) = \mathcal{W}_1(\varepsilon, x; X) + \mathcal{W}_2(\varepsilon, x; X) \tag{B.15}$$

where

$$\mathcal{W}_1(\varepsilon, x; X) = (Z^o(X) - 1)\left(\int_0^{\varepsilon} e^{-\lambda_1\tau}u(x)d\tau + \int_{\varepsilon}^{+\infty} e^{-\lambda_1\tau}u(\sigma^o(\hat{X}(x,\varepsilon,\tau;X)))d\tau\right) \tag{B.16}$$

and

$$\mathcal{W}_2(\varepsilon, x; X) = \int_0^{\varepsilon} e^{-\lambda_0\tau}\left(1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(X + xs)e^{\Delta s}ds\right)u(x)d\tau \tag{B.17}$$

$$+\int_{\varepsilon}^{+\infty} e^{-\lambda_0\tau}\left(1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(\hat{X}(x,\varepsilon,\tau;X))e^{\Delta s}ds\right)u(\sigma^o(\hat{X}(x,\varepsilon,\tau;X)))d\tau.$$

From (B.16), we deduce

$$\frac{\partial\mathcal{W}_1}{\partial\varepsilon}(0,x,X) = (Z^o(X) - 1)\Bigg(u(x) - u(\sigma^o(X)) \tag{B.18}$$

$$+\int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial\hat{X}}{\partial\varepsilon}(x,\varepsilon,s;X)|_{\varepsilon=0}d\tau\Bigg).$$

Using (B.2), this expression can be simplified as

$$\frac{\partial\mathcal{W}_1}{\partial\varepsilon}(0,x,X) = (Z^o(X) - 1)\Bigg(u(x) - u(\sigma^o(X)) \tag{B.19}$$

$$+\left(\frac{x}{\sigma^o(X)} - 1\right)\int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial\tau}(\tau;X)d\tau\Bigg).$$

Integrating by parts, we also have

$$\text{(B.20)}\quad \int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial\tau}(\tau;X)d\tau$$

$$= \left[e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty} + \lambda_1\int_0^{+\infty} e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))d\tau.$$

$$= -u(\sigma^o(X)) + \lambda_1\varphi^o(X) = \sigma^o(X)\dot{\varphi}^o(X)$$

where the last equality follows from (B.10). Inserting into (B.19) yields

$$\text{(B.21)}\quad \frac{\partial\mathcal{W}_1}{\partial\varepsilon}(0,x,X) = (Z^o(X)-1)\left(u(x) - u(\sigma^o(X)) + (x-\sigma^o(X))\,\dot{\varphi}^o(X)\right).$$

From (B.17) and (5.10), we deduce

$$\text{(B.22)}\quad \frac{\partial\mathcal{W}_2}{\partial\varepsilon}(0,x,X) = u(x) - u(\sigma^o(X))$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial\hat{X}}{\partial\varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0}d\tau$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial\hat{X}}{\partial\varepsilon}(x,\varepsilon,s;X)|_{\varepsilon=0}e^{\Delta s}ds\right)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Using (B.2), this expression can be simplified as

$$\text{(B.23)}\quad \frac{\partial\mathcal{W}_2}{\partial\varepsilon}(0,x,X) = u(x) - u(\sigma^o(X))$$

$$+\left(\frac{x}{\sigma^o(X)}-1\right)\left(\int_0^{+\infty} e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial\tau}(\tau;X)d\tau\right.$$

$$\left.+\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial\tau}(s;X)e^{\Delta s}ds\right)u(\sigma^o(X^o(\tau;X)))d\tau\right).$$

Differentiating (E.8) with respect to X and using (B.1) yields

$$\text{(B.24)}\quad \sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) = \sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau} - \Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial s}(s;X)e^{\Delta s}ds.$$

Using (B.24), we now rewrite

$$\text{(B.25)}\quad \int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial\tau}(s;X)e^{\Delta s}ds\right)u(\sigma^o(X^o(\tau;X)))d\tau$$

$$= \int_0^{+\infty} e^{-\lambda_0\tau}\left(\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) - \sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Integrating by parts, we also have

$$\text{(B.26)}\quad \int_0^{+\infty} e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial\tau}(\tau;X)d\tau$$

$$= \left[e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty} +$$

$$\int_0^{+\infty}\left(\lambda_0\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right) - \frac{\partial Z}{\partial\tau}(\tau;X) - \Delta(Z^o(X)-1)e^{-\Delta\tau}\right)e^{-\lambda_0\tau}u(\sigma^o(X^o(\tau;X)))d\tau.$$

$$= \lambda_0 \mathcal{W}^o(X) - u(\sigma^o(X)) - \lambda_1 (Z^o(X) - 1)\varphi^o(X) - \int_0^{+\infty} e^{-\lambda_0 \tau} \frac{\partial Z}{\partial \tau}(\tau; X) u(\sigma^o(X^o(\tau; X)))d\tau.$$

Using (B.25) and (B.26) and inserting into (B.23) yields

$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X))$$

$$+ \left(\frac{x}{\sigma^o(X)} - 1\right) \Bigg(\lambda_0 \mathcal{W}^o(X) - u(\sigma^o(X)) - \lambda_1 (Z^o(X) - 1)\varphi^o(X)$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) - \frac{\partial Z}{\partial \tau}(\tau; X) - \sigma^o(X) \dot{Z}^o(X) e^{-\Delta \tau} \right) u(\sigma^o(X^o(\tau; X)))d\tau \Bigg).$$

Using (B.3) and simplifying yields

$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \tag{B.27}$$

$$+ \left(\frac{x}{\sigma^o(X)} - 1\right) \Bigg(\lambda_0 \mathcal{W}^o(X) - Z^o(X) u(\sigma^o(X)) + (Z^o(X) - 1) u(\sigma^o(X)) - \sigma^o(X) \dot{Z}^o(X) \varphi^o(X)$$

$$- \lambda_1 (Z^o(X) - 1)\varphi^o(X) \Bigg).$$

Using (B.13) and (B.10) and simplifying yields

$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) + (x - \sigma^o(X)) \left(\dot{\mathcal{W}}^o(X) - (Z^o(X) - 1)\dot{\varphi}^o(X) - \dot{Z}^o(X)\varphi^o(X) \right). \tag{B.28}$$

Gathering (B.28) and (B.21) finally yields

$$\frac{\partial \mathcal{W}}{\partial \varepsilon}(0, x, X) = Z^o(X) \Big(u(x) - u(\sigma^o(X)) \Big) + (x - \sigma^o(X)) \left(\dot{\mathcal{W}}^o(X) - \dot{Z}^o(X)\varphi^o(X) \right).$$

Because $\frac{\partial \mathcal{W}}{\partial \varepsilon}(0, x, X)$ so obtained is strictly concave in x, the following first-order condition is necessary and sufficient for an interior optimum obtained from (B.5) and (B.6):

$$0 = \frac{\partial^2 \mathcal{W}}{\partial \varepsilon \partial x}(0, \sigma^o(X), X)$$

Developing, we find

$$\sigma^o(X) = \zeta + \frac{\dot{\mathcal{W}}^o(X)}{Z^o(X)} - \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X). \tag{B.29}$$

which writes as (5.16).

Inserting (5.16) into (B.9), we now obtain

$$\sigma^o(X) \left(\sigma^o(X) - \zeta + \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right) = \lambda_0 \mathcal{V}^o(X) - \lambda_1 \mathcal{V}_\infty + \frac{1}{2}(\sigma^o(X) - \zeta)^2.$$

Simplifying, we obtain

$$\left(\sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2 = 2\lambda_0 \mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2. \tag{B.30}$$

Taking then the highest root to (B.30), we obtain

$$\sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X) = \sqrt{2\lambda_0\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2}. \tag{B.31}$$

Inserting (5.16) into (B.31) and simplifying finally yields (5.14).
LIMITING BEHAVIOR. From (E.8) and the fact that $X^o(\tau; X) \geq \overline{X}$ for all $\tau \geq 0$ and $X \geq \overline{X}$, it follows that

$$Z(\tau; X) = Z^o(\overline{X})e^{-\Delta\tau} \quad \forall \tau \geq 0, X \geq \overline{X}. \tag{B.32}$$

Inserting into (5.6) immediately yields (5.15). From there, it immediately follows that

$$\sigma^o(X) = \zeta \quad \forall X \geq \overline{X}. \tag{B.33}$$

Q.E.D.

PROOF OF PROPOSITION 5: Clearly (5.23) holds for $X \geq \overline{X}$. We turn to the more difficult case, $X \in [0, \overline{X})$. Consider the pair $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with a belief index $Z^o(X)$ now defined as

$$\sigma^e(X, Z^o(X))\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \tag{B.34}$$

with the boundary condition

$$Z^o(0) = 1. \tag{B.35}$$

Observe that, provided that $\sigma^e(X, Z)$ remains positive, such a $Z^o(X)$ is uniquely defined and satisfies the same properties as in Lemma B.3. In particular, $Z^o(X)$ is positive for all $X \in [0, \overline{X})$.

We shall prove that $\mathcal{V}^e(X, Z^o(X) \equiv \mathcal{V}^o(X)$, $\sigma^e(X, Z^o(X)) \equiv \sigma^o(X)$ and $Z^o(X)$ as defined above altogether form a *SME*. To ease notations, define accordingly $\mathcal{W}^o(X)$ as in (B.7).

First, notice that, from (A.24), it immediately follows that, for $X \in [0, \overline{X})$,

$$\lambda_0\mathcal{W}^e(X, Z^o(X)) = \sup_{x\in\mathcal{X}} \left\{ Z^o(X)u(x) + x\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \Delta(1 - F(X) - Z^o(X))\frac{\partial \mathcal{W}^e}{\partial Z}(X, Z^o(X)) \right\} \tag{B.36}$$

where we remind that $\mathcal{W}^e(X, Z^o(X)) = Z^o(X)\mathcal{V}^e(X, Z^o(X))$.

Using (A.28) and (B.34), we rewrite (B.36) as

$$\lambda_0\mathcal{W}^e(X, Z^o(X)) = \sup_{x\in\mathcal{X}} \left\{ Z^o(X)u(x) + x\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \sigma^e(X, Z^o(X))\dot{Z}^o(X)\varphi^e(X, Z^o(X)) \right\} \tag{B.37}$$

where the maximand above is achieved for

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)}\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) \quad \forall X \in [0, \overline{X}). \tag{B.38}$$

Still using (A.28), we obtain the following expression of the total derivative of $\mathcal{W}^e(X, Z^o(X))$

$$\frac{d\mathcal{W}^e}{dX}(X, Z^o(X)) = \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \dot{Z}^o(X)\varphi^e(X, Z^o(X)) \quad \forall X \in [0, \overline{X}). \tag{B.39}$$

Inserting (B.39) into (B.38) yields

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)}\left(\frac{d}{dX}\mathcal{W}^e(X, Z^o(X)) - \dot{Z}^o(X)\varphi^e(X, Z^o(X))\right) \quad \forall X \in [0, \overline{X}). \tag{B.40}$$

Also, (A.19) allows us to rewrite

$$\text{(B.41)}\qquad \varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^e(\widetilde{X}^e(\tau; X, Z^o(X)), \widetilde{Z}^e(\tau; X, Z^o(X))))d\tau.$$

At equilibrium, DM expects that the feedback rule $\sigma^o(X') = \sigma^e(X', Z^o(X'))$ prevails for all $X' > X$ and in particular for $X' = X^o(\tau; X)$ for $\tau > 0$, Observe that the future trajectory of stock and beliefs is thus such that $\widetilde{X}^e(\tau; X, Z^o(X)) = X^o(\tau; X)$ and $\widetilde{Z}^e(\tau; X, Z^o(X))) = Z^o(X^o(\tau; X))$ for all $\tau > 0$. Hence, we rewrite (B.41) as

$$\varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^e(X^o(\tau; X), Z^o(X^o(\tau; X)))d\tau$$

or

$$\text{(B.42)}\qquad \varphi^o(X) = \varphi^e(X, Z^o(X)).$$

Inserting (B.42) into (B.40) yields

(B.43)

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)} \left(Z^o(X) \frac{d}{dX} \mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X)(\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X)) \right) \quad \forall X \in [0, \overline{X}).$$

Rewriting (B.37), we obtain that $\mathcal{V}^e(X, Z^o(X))$ solves

$$\text{(B.44)}\qquad \lambda_0 Z^o(X)\mathcal{V}^e(X, Z^o(X)) = \sup_{x\in\mathcal{X}} Z^o(X)u(x)$$

$$+x\left(Z^o(X) \frac{d}{dX} \mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X)(\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X)) \right) + \sigma^e(X, Z^o(X))\dot{Z}^o(X)\varphi^o(X)$$

where the maximum is achieved with $\sigma^e(X, Z^o(X))$ that satisfies (B.43).

From this, we now observe that $\mathcal{V}^o(X) \equiv \mathcal{V}^e(X, Z^o(X)$ and $\sigma^o(X) = \sigma^e(X, Z^o(X))$ altogether solve

$$\text{(B.45)}\qquad \lambda_0 Z^o(X)\mathcal{V}^o(X) = \sup_{x\in\mathcal{X}} Z^o(X)u(x) + x\Big(Z^o(X)\dot{\mathcal{V}}^o(X) + \dot{Z}^o(X)(\mathcal{V}^o(X) - \varphi^o(X))\Big)$$

$$+\sigma^o(X)\dot{Z}^o(X)\varphi^o(X)$$

where $\sigma^o(X)$, which achieves the maximum on the r.-h.s. above, satisfies

$$\text{(B.46)}\qquad \sigma^o(X) = \zeta + \frac{1}{Z^o(X)} \Big(Z^o(X)\dot{\mathcal{V}}^o(X) + \dot{Z}^o(X)(\mathcal{V}^o(X) - \varphi^o(X)) \Big) \quad \forall X \in [0, \overline{X}).$$

Inserting (B.46) into (B.45), rearranging and simplifying yields that $\mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X))$ indeed satisfies (5.14) as requested with any (continuously differentiable) *SME*. Moreover, and from (4.14), the boundary condition (5.15) holds. Hence, $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with the associated index $Z^o(X)$ that satisfies (B.34)-(B.35) form a *SME*. *Q.E.D.*

Bounds. This implementation of the optimum is useful to get bounds on payoffs and actions at the optimum. Proposition B.1 below provides tight bounds on the *Stock-Markov* value function and the feedback rule for any SME, and in particular the one, described in Section 5, that implements the optimal trajectory.

Proposition B.1 *$\mathcal{V}^o(X)$, $\varphi^o(X)$ and $\sigma^o(X)$ admit the following bounds:*

$$\text{(B.47)}\qquad \varphi^o(X) \le \mathcal{V}_\infty \le \mathcal{V}^o(X) \le \mathcal{V}_\infty \left(1 + \frac{\Delta}{\lambda_0}(1 - F(X)) \right) \quad \forall X \in \left[0, \overline{X}\right],$$

$$\text{(B.48)}\qquad \zeta\sqrt{\frac{\lambda_0}{\lambda_1}} \le \sigma^o(X) \le \zeta \quad \forall X \in \left[0, \overline{X}\right].$$

PROOF OF PROPOSITION B.1: First, using (E.8) and noticing that $F(X) \leq F(X^o(\tau;X)) \leq 1$ for $\tau \geq 0$, we obtain the bounds

$$Z^o(X)e^{\Delta\tau} \leq Z(\tau;X) = Z^o(X^o(\tau;X)) \leq 1 - F(X) + F(X)e^{-\Delta\tau} \quad \forall \tau \geq 0, X \geq 0. \tag{B.49}$$

Inserting into the definition of $\mathcal{V}^o(X)$ given in (5.6) and integrating, we obtain

$$Z^o(X)\varphi^o(X) \leq Z^o(X)\mathcal{V}^o(X) \leq (1 - F(X))\frac{\lambda_1}{\lambda_0} + F(X)\varphi^o(X) \quad \forall X \geq 0. \tag{B.50}$$

Of course, we have

$$\varphi^o(X) \leq \mathcal{V}_\infty \quad \forall X \geq 0 \tag{B.51}$$

which is the l.-h.s. inequality in (B.47). Inserting into (B.50) yields the r.-h.s. inequality in (B.47). The second inequality immediately follows from (5.23) and (A.29) taken for $Z = Z^o(X)$.

To obtain the r.-h.s. inequality in (B.48), first observe that (4.12), (4.15) and (5.23) imply

$$\sigma^o(X) \leq \sqrt{2\lambda_1\mathcal{V}_\infty} = \zeta$$

as requested. To obtain the l.-h.s. inequality in (B.48), observe that $\dot{Z}^o(X) \leq 0$ (from Lemma B.3) and $\varphi^o(X) \geq 0$ altogether imply

$$\sigma^o(X) \geq \sqrt{2\lambda_0\mathcal{V}^o(X)}.$$

Using the second left inequality in (B.47) yields the result. *Q.E.D.*

APPENDIX C: *SME* WITH NON-OBSERVABLE IMPULSE DEVIATIONS

PROOF OF PROPOSITION 6: Being given that each decision-maker takes as given the evolution of beliefs when looking for an optimal action, $\mathcal{V}^{no}(X)$ as defined by (6.4) and following Definition 2 solves

$$\mathcal{V}^{no}(X) = \sup_{\mathcal{A}} \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \sigma^{no}(X^{no}(s;X))\frac{\dot{Z}^{no}(X^{no}(s;X))}{Z^{no}(X^{no}(s;X))}\right)ds} u(\sigma^{no}(X^{no}(\tau;X)))d\tau. \tag{C.1}$$

where $Z^{no}(X)$ is consistent with the feedback rule $\sigma^{no}(X)$ that is optimal for problem (C.1) and satisfies (6.1)-(6.2).

Let first define

$$\mathcal{W}^{no}(X) = Z^{no}(X)\mathcal{V}^{no}(X). \tag{C.2}$$

It is routine to show that, at any point of differentiability, $\mathcal{W}^{no}(X)$ satisfies the following Hamilton-Bellman-Jacobi equation for problem (6.4):

$$\lambda_0\mathcal{W}^{no}(X) = \max_{x\in\mathcal{X}} Z^{no}(X)u(x) + x\dot{\mathcal{W}}^{no}(X). \tag{C.3}$$

The maximand is obtained for an interior solution

$$\sigma^{no}(X) = \zeta + \frac{\dot{\mathcal{W}}^{no}(X)}{Z^{no}(X)}. \tag{C.4}$$

Simplifying yields (6.10). Inserting (C.4) into (C.3) yields

$$\lambda_0\mathcal{W}^{no}(X) = Z^{no}(X)\lambda_1\mathcal{V}_\infty + \frac{(\dot{\mathcal{W}}^{no}(X))^2}{2Z^{no}(X)} + \zeta\dot{\mathcal{W}}^{no}(X).$$

Solving this second-degree equation in $\dot{\mathcal{W}}^{no}(X)$ yields

$$\dot{\mathcal{W}}^{no}(X) = Z^{no}(X)\left(-\zeta + \sqrt{2\lambda_0\frac{\mathcal{W}^{no}(X)}{Z^{no}(X)}}\right). \tag{C.5}$$

Rewriting this condition in terms of $\mathcal{V}^{no}(X)$ yields (6.8).

The boundary condition (6.9) is immediate. For future reference, observe that it also writes in terms of $\mathcal{W}^{no}(X)$ as

$$\mathcal{W}^{no}(X) = \mathcal{Z}^{no}(X)\mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{C.6}$$

Q.E.D.

Existence. Finally, our last result proves existence of a SME with non-observable impulse deviations. Its proof consists in studying the properties of the system of first-order differential equations satisfied by $(\mathcal{V}^{no}(X), Z^{no}(X))$ and showing that the boundary conditions at $X = 0$ and $X = \overline{X}$ for that system are satisfied.

Proposition C.1 *A Stock-Markov value function with non-observable deviations $\mathcal{V}^{no}(X)$ and an associated feedback rule $\sigma^{no}(X)$ always exist.*

Proof of Proposition C.1: We consider the flow of the differential system made of (6.1) and (C.5) with the initial condition for $\mathcal{Z}^{no}(X)$ given by (6.2) together with an arbitrary initial condition for $\mathcal{W}^{no}(X)$ given by

$$\mathcal{W}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right]. \tag{C.7}$$

We look for such an initial value $\mathcal{W}^{no}(0)$ so that the terminal condition (C.6) is satisfied.

Observe that the system (6.1)-(C.5) is Lipschitz-continuous on the open domain

$$\mathcal{W}^{no}(X) > 0 \tag{C.8}$$

We now define $\widetilde{\mathcal{W}}^{no}(Y) = \mathcal{W}^{no}(X)$, $Z^{no}(Y) = Z^{no}(X)$, $\widetilde{\sigma}^{no}(Y) = \sigma^{no}(X)$ where $Y = 1-F(X) \in [0,1]$. Let also denote $R(Y) = f(F^{-1}(1-Y))$ for all $Y \in [0,1]$. First, notice that we also have $\dot{Z}^{no}(Y) = -\frac{\dot{\mathcal{Z}}^{no}(X)}{R(Y)}$ and $\dot{\widetilde{\mathcal{W}}}^{no}(Y) = -\frac{\dot{\mathcal{W}}^{no}(X)}{R(Y)}$. Second, using (6.10) and (C.2), we rewrite

$$\widetilde{\sigma}^{no}(Y) = \sqrt{2\lambda_0 \frac{\widetilde{\mathcal{W}}^{no}(Y)}{\mathcal{Z}^{no}(Y)}}. \tag{C.9}$$

We now transform the system of first-order differential equations (6.1)-(C.5) as

$$\dot{\widetilde{\mathcal{W}}}^{no}(Y) = \frac{Z^{no}(Y)}{R(Y)}(\zeta - \widetilde{\sigma}^{no}(Y)), \tag{C.10}$$

$$\dot{Z}^{no}(Y) = \frac{\Delta(Z^{no}(Y) - Y)}{R(Y)\widetilde{\sigma}^{no}(Y)}. \tag{C.11}$$

together with the following boundary conditions

$$\widetilde{\mathcal{W}}^{no}(1) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z^{no}(1) = 1 \tag{C.12}$$

and

$$\widetilde{\mathcal{W}}^{no}(0) = Z^{no}(0)\mathcal{V}_\infty. \tag{C.13}$$

Satisfying boundary conditions at the two end-points $Y = 0$ and $Y = 1$ requires a global analysis of the system. The first step consists in observing that the new system (C.10) can be transformed into an homogenous system expressed in terms of a variable $\tau \in \mathbb{R}_+$ such that (slightly abusing notations by not changing the names of variables although they now depend on τ)

$$\dot{\widetilde{\mathcal{W}}}^{no}(\tau) = Z^{no}(\tau)(-\zeta + \widetilde{\sigma}^{no}(\tau)), \tag{C.14}$$

$$\dot{Z}^{no}(\tau) = \frac{\Delta(Y - Z^{no}(\tau))}{\widetilde{\sigma}^{no}(Y)}, \tag{C.15}$$

$$\dot{Y}(\tau) = -R(Y(\tau)) \tag{C.16}$$

together with the following boundary conditions

$$\widetilde{\mathcal{W}}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z^{no}(0) = 1, \quad Y(0) = 1 \tag{C.17}$$

and

$$\lim_{\tau \to +\infty} \widetilde{\mathcal{W}}^{no}(\tau) - Z^{no}(\tau)\mathcal{V}_\infty = 0, \quad \lim_{\tau \to +\infty} Y(\tau) = 0. \tag{C.18}$$

Observe that $Y(\tau)$ is decreasing. Moreover, direct integration of (C.16) together with the third condition in (C.17) yields

$$\tau = \int_{Y(\tau)}^{1} \frac{dY}{R(Y)}. \tag{C.19}$$

Consider now the hyperplans

$$\mathcal{D}_0 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} = \frac{\lambda_1 \mathcal{V}_\infty}{\lambda_0} Z\right\} \text{ and } \mathcal{D}_1 = \left\{(0, Z, Y) \in {\mathbb{R}_+}^3\right\}.$$

Observe that the segment for initial conditions

$$\mathcal{D}_3 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z = 1, \quad Y = 1\right\}$$

lies in the cone of the positive orthant whose faces are the hyperplans $\mathcal{D}_0$ and $\mathcal{D}_1$. Observe that the hyperplan

$$\mathcal{D}_4 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} = Z\mathcal{V}_\infty\right\}$$

belongs to that cone since $0 < \mathcal{V}_\infty < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ and intersects $\mathcal{D}_0$ and $\mathcal{D}_1$ at the origin only.

Condition (C.19) shows that any trajectory is such that $Y(\tau)$ is decreasing and remains in the bandwith

$$\mathcal{D}_2 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } Y \in [0, 1]\right\}.$$

Moreover, Condition (C.19) also implies that a trajectory reaches $Y = 0$ in finite time if and only if $\int_0^1 \frac{dY}{R(Y)} < +\infty$. If instead $\int_0^1 \frac{dY}{R(Y)} = +\infty$, $Y = 0$ is only reached asymptotically.

Note that any solution to the system (C.14)-(C.15)-(C.16) with initial conditions (C.17) that would cross the hyperplan $\mathcal{D}_0$ at a time $\overline{T}$ crosses it from below (from the fact that $\dot{\widetilde{\mathcal{W}}}^{no}(\overline{T}) \leq 0$ and that direction is not in the hyperplan $\mathcal{D}_0$). Similarly, any solution to the system (C.14)-(C.15)-(C.16) with initial conditions (C.17) that would cross the hyperplan $\mathcal{D}_1$ at a time τ_1 reaches it from above (from the fact that $\dot{Z}^{no}(\tau_1) = +\infty$ and that direction is not in the hyperplan $\mathcal{D}_1$). Moreover, such trajectory stops there.

Because the system is continuous on the open positive cone defined by the faces $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$, any trajectory starting from the segment $\mathcal{D}_3$ can be extended till it reaches the boundaries of this domain in finite time (Nemytskii and Stepanov, 1989, p. 307). Because the flow of the system is continuous, the image of $\mathcal{D}_3$ which is connected and compact consists of a continuous line $\mathcal{L}$ that might lie on $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$. Observe that, for the initial condition $\widetilde{\mathcal{W}}(0) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$, the trajectory immediately crosses $\mathcal{D}_0$ and goes out of the cone. Similarly, for the initial condition $\widetilde{\mathcal{W}}(0) = \frac{D}{\lambda_0}$, the trajectory immediately reaches $\mathcal{D}_1$ and stays there. By continuity of the flow of the differential system, trajectories with an initial condition $\widetilde{\mathcal{W}}(0)$ in a neighborhood of $\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ goes through $\mathcal{D}_0$ while trajectories with an initial condition $\widetilde{\mathcal{W}}(0)$ in a neighborhood of $\frac{D}{\lambda_0}$ reaches $\mathcal{D}_1$. Two cases may *a priori* arise. First, $\mathcal{L}$ may not go though the origin $(0,0,0)$. In this case, and by continuity, the part of $\mathcal{L}$ that lies on $\mathcal{D}_2$ necessarily crosses $\mathcal{D}_4$ somewhere and the boundary problem has a solution such that $\lim_{\tau\to+\infty} \widetilde{\mathcal{W}}(\tau) = \lim_{\tau\to+\infty} Z^{no}(\tau)\mathcal{V}_\infty > 0$ or, expressed in terms of original variables $\mathcal{W}^{no}(\overline{X}) = Z^{no}(\overline{X})\mathcal{V}_\infty > 0$. Second, $\mathcal{L}$ may go though the origin $(0,0,0)$. In this case, there is a trajectory that satisfies the boundary condition with $\lim_{\tau\to+\infty} \widetilde{\mathcal{W}}(\tau) = \lim_{\tau\to+\infty} Z^{no}(\tau)\mathcal{V}_\infty = 0$ or expressed in terms of original variables $\mathcal{W}^{no}(\overline{X}) = Z^{no}(\overline{X})\mathcal{V}_\infty = 0$.

Q.E.D.

APPENDIX D: RUNNING EXAMPLE

Proof of Proposition 3: Observe that (4.7) rewrites now as

$$Z(\tau) = -(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}. \tag{D.1}$$

It is straightforward to check that $Z(\tau) \geq 1-q$ for all $\tau > 0$ when $Z \geq 1-q$. Since the optimal trajectory starts from $Z = 1$, this condition always holds.

This expression of $Z(\tau)$ allows us to rewrite the definition (4.11) for $\mathcal{V}^e(X,Z)$ in a quasi-explicit form as

$$\begin{aligned} Z\mathcal{V}^e(X,Z) = \max_{\mathbf{x},\overline{T}} \int_0^{\overline{T}} e^{-\lambda_0\tau}\left(-(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}\right) u(x(\tau))d\tau \\ + e^{-\lambda_0\overline{T}}\left(-(1-Z)e^{-\Delta\overline{T}} + 1 - q + qe^{-\Delta\overline{T}}\right)\mathcal{V}_\infty \end{aligned} \tag{D.2}$$

subject to

$$\int_0^{\overline{T}} x(\tau)d\tau = \overline{X} - X. \tag{D.3}$$

Solving this problem is straightforward. Let denote by μ the multiplier for (D.3). We form the Lagrangean

$$\begin{aligned} \mathcal{L}(\mathbf{x},\overline{T}) = \int_0^{\overline{T}} e^{-\lambda_0\tau}\left(-(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}\right) u(x(\tau))d\tau \\ + e^{-\lambda_0\overline{T}}\left(-(1-Z)e^{-\Delta\overline{T}} + 1 - q + qe^{-\Delta\overline{T}}\right)\mathcal{V}_\infty + \mu\left(\overline{X} - X - \int_0^{\overline{T}} x(\tau)d\tau\right). \end{aligned}$$

Pointwise optimization for this strictly concave objective yields the following expression of the optimal action at any point in time

$$\zeta - x^e(\tau) = \frac{\mu e^{\lambda_0\tau}}{Z(\tau)} \tag{D.4}$$

where, for simplicity, we omit the dependence on the state variables (X,Z).

Integrating over $\left[0,\overline{T}^e\right]$ yields

$$\zeta\overline{T}^e - (\overline{X} - X) = \mu\int_0^{\overline{T}^e} \frac{e^{\lambda_0\tau}}{Z(\tau)}d\tau. \tag{D.5}$$

Optimizing now with respect to $\overline{T}$ and assuming the quasi-concavity of the objective in $\overline{T}$ yields the following necessary first-order condition

$$e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e) u(x^e(\overline{T}^{e-})) + \mathcal{V}_\infty e^{-\lambda_0 \overline{T}^e} \left(-\lambda_0 Z(\overline{T}^e) + \dot{Z}(\overline{T}^e)\right) = \mu x^e(\overline{T}^{e-})$$

where $x^e(\overline{T}^{e-})$ denotes the l.h.-s limit of $x^e(\tau)$ at $\overline{T}^e$. Simplifying, we get

$$\zeta x^e(\overline{T}^{e-}) - \frac{(x^e(\overline{T}^{e-}))^2}{2} + \mathcal{V}_\infty \left(-\lambda_0 + \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right) = \mu \frac{e^{-\lambda_0 \overline{T}^e}}{Z(\overline{T}^e)} x^e(\overline{T}^{e-})$$

Using (D.4) taken at $\tau = \overline{T}^e$, we rewrite the r.-h.s. and get

$$\zeta x^e(\overline{T}^{e-}) - \frac{(x^e(\overline{T}^{e-}))^2}{2} + \mathcal{V}_\infty \left(-\lambda_0 + \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right) = x^e(\overline{T}^{e-})(\zeta - x^e(\overline{T}^{e-}))$$

Simplifying further yields

$$x^e(\overline{T}^{e-}) = \zeta \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}.$$

From (D.4) taken at $\tau = \overline{T}^e$, we then get

$$\text{(D.6)} \qquad \mu \frac{e^{\lambda_0 \overline{T}^e}}{Z(\overline{T}^e)} = \zeta \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right).$$

Inserting (D.6) into (D.5) and (D.4) finally yields (D.7) and (D.8) respectively:

$$\text{(D.7)} \qquad \overline{T}^e = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right) e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e) \int_0^{\overline{T}^e} \frac{e^{\lambda_0 \tau}}{Z(\tau)} d\tau,$$

$$\text{(D.8)} \qquad x^e(\tau) = \zeta \left(1 - e^{-\lambda_0(\overline{T}^e - \tau)} \frac{Z(\overline{T}^e)}{Z(\tau)} \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\right) \quad \forall \tau \in [0, \overline{T}^e).$$

Specializing this solution to the case $X = 0$ and $Z = 1$ yields the optimal trajectory described in (4.24) and (4.22) with $Z(\tau)$ being given by (4.23). Because $\frac{e^{\lambda_0 \tau}}{Z(\tau)}$ is increasing, $x^e(\tau)$ is itself decreasing over $[0, \overline{T}^e)$.

Specializing further to the case $q = 0$ yields the optimal trajectory when the tipping point is known being at $\overline{X}$ for sure. In this case, $\overline{T}^k$ is given by (4.20) while the optimal action is now

$$\text{(D.9)} \qquad x^k(\tau) = \begin{cases} \zeta \left(1 - e^{-\lambda_0(\overline{T}^k - \tau)} \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right)\right) < \zeta & \text{for } \tau \in [0, \overline{T}^k), \\ \zeta & \text{for } \tau \geq \overline{T}^k. \end{cases}$$

Because $Z(\tau)$ is decreasing, one has

$$\overline{T}^k < \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) e^{-\lambda_0 \overline{T}^k} \int_0^{\overline{T}^k} e^{\lambda_0 \tau} d\tau = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 \overline{T}^k}}{\lambda_0}.$$

Consider now the function $\delta(t) \equiv t - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 t}}{\lambda_0}$. We have $\delta(T^k) = T^m$, $\delta(0) = 0$ and $\delta'(t) = 1 - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) e^{-\lambda_0 t} > 0$. Hence, there is a unique positive root $0 < \overline{T}^k < T^m$ for (4.22). *Q.E.D.*

Proof of Proposition 7: The equilibrium trajectory starting from $X = 0$ solves

$$\max_{\mathbf{x},X(\cdot),\overline{T}} \int_0^{\overline{T}} e^{-\lambda_0\tau} Z^{no}(X(\tau))u(x(\tau))d\tau + e^{-\lambda_0\overline{T}} Z^{no}(\overline{X})\mathcal{V}_\infty$$

subject to (4.5), $X(0) = X$, and $X(T) = \overline{X}$,

where $Z^{no}(X)$ is given by (6.1) and (6.2).

Let denote by μ the costate variable for (4.5). The Hamiltonian for this control problem is

$$\mathcal{H}^{no}(X, x, \tau, \lambda) = e^{-\lambda_0\tau} Z^{no}(X)u(x) + \mu x. \tag{D.10}$$

The *Maximum Principle* with free final time and scrap value gives us the following necessary conditions for an optimal arc $(X^{no}(\tau), x^{no}(\tau), \overline{T}^{no})$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)

Costate variable. $\mu(\tau)$ is continuously differentiable on $\mathbb{R}_+$ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^{no}}{\partial X}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau))$$

or

$$-\dot{\mu}(\tau) = e^{-\lambda_0\tau} \dot{Z}^{no}(X^{no}(\tau))u(x^{no}(\tau)) \quad \forall \tau \in \left[0, \overline{T}^{no}\right]. \tag{D.11}$$

Transversality conditions. The boundary conditions $X^{no}(0) = 0$ and $X^{no}(\overline{T}^{no}) = \overline{X}$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \overline{T}^{no}$.

Control variable $x^{no}(\tau)$.

$$x^{no}(\tau) \in \arg\max_{x \geq 0} \mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau)).$$

Because $\mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}^{no}}{\partial x}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau)) = 0$$

or

$$x^{no}(\tau) = \zeta + e^{\lambda_0\tau} \frac{\mu(\tau)}{Z^{no}(X^{no}(\tau))}. \tag{D.12}$$

Free-end point conditions. The optimality condition with respect to $\overline{T}$ writes as

$$\mathcal{H}^{no}(X^{no}(\overline{T}^{no}), x^{no}(\overline{T}^{no}), \overline{T}^{no}, \mu(\overline{T}^{no})) - \lambda_0 Z^{no}(\overline{X}) e^{-\lambda_0\overline{T}^{no}} \mathcal{V}_\infty = 0. \tag{D.13}$$

From (D.12), we get

$$x^{no}(\overline{T}^{no}) = \zeta + e^{\lambda_0\overline{T}^{no}} \frac{\mu(\overline{T}^{no})}{Z^{no}(\overline{X})}. \tag{D.14}$$

Using (D.10), (D.14), inserting into (D.13) and simplifying yields

$$\zeta x^{no}(\overline{T}^{no-}) - \frac{1}{2}\left(x^{no}(\overline{T}^{no-})\right)^2 - \lambda_0\mathcal{V}_\infty = x^{no}(\overline{T}^{no-})(\zeta - x^{no}(\overline{T}^{no-}))$$

or

$$x^{no}(\overline{T}^{no-}) = \zeta\sqrt{\frac{\lambda_0}{\lambda_1}}. \tag{D.15}$$

where, to account for the discontinuity in action at $\overline{T}^{no}$, we denote by $x^{no}(\overline{T}^{no-})$ the l.-h. side limit of $x^{no}(\tau)$ as $\tau \to \overline{T}^{no-}$.

Characterization. Using (D.1) for the optimal arc starting from $Z = 1$, we get

$$Z(\tau) = 1 - q + qe^{-\Delta\tau}. \tag{D.16}$$

Along the trajectory, we must have

$$Z^{no}(X^{no}(\tau)) = Z(\tau) \quad \forall \tau \le \overline{T}^{no}. \tag{D.17}$$

Differentiating, we get

$$\dot{Z}^{no}(X^{no}(\tau)) = \frac{\dot{Z}(\tau)}{x^{no}(\tau)} = -\frac{q\Delta e^{-\Delta\tau}}{x^{no}(\tau)} \tag{D.18}$$

Now, we rewrite (D.12) as

$$\mu(\tau) = Z^{no}(X^{no}(\tau))(x^{no}(\tau) - \zeta)e^{-\lambda_0\tau}.$$

Differentiating w.r.t. τ and using (D.18) yields the following ordinary differential equation for $x^{no}(\tau)$:

$$\dot{x}^{no}(\tau) - \left(\lambda_0 - \frac{\dot{Z}(\tau)}{2Z(\tau)}\right) x^{no}(\tau) = -\lambda_0\zeta.$$

It is routine to check that the solution of this ordinary differential equation is of the form

$$x^{no}(\tau) = \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}\left(C_0 - \lambda_0\zeta\int_0^\tau e^{-\lambda_0 s}\sqrt{Z(s)}ds\right) \tag{D.19}$$

for some constant C_0. Using (D.15), this constant is determined as

$$\zeta\sqrt{\frac{\lambda_0}{\lambda_1}} = \frac{e^{\lambda_0\overline{T}^{no}}}{\sqrt{Z(\overline{T}^{no})}}\left(C_0 - \lambda_0\zeta\int_0^{\overline{T}^{no}} e^{-\lambda_0 s}\sqrt{Z(s)}ds\right)$$

or

$$C_0 = \zeta\sqrt{\frac{\lambda_0}{\lambda_1}}e^{-\lambda_0\overline{T}^{no}}\sqrt{Z(\overline{T}^{no})} + \lambda_0\zeta\int_0^{\overline{T}^{no}} e^{-\lambda_0 s}\sqrt{Z(s)}ds. \tag{D.20}$$

Integrating (D.19), the corresponding stock evolves according to

$$X^{no}(\tau) = C_0\int_0^\tau \frac{e^{\lambda_0 s}}{\sqrt{Z(s)}}ds - \lambda_0\zeta\int_0^\tau \frac{e^{\lambda_0 s}}{\sqrt{Z(s)}}\left(\int_0^s e^{-\lambda_0 s'}\sqrt{Z(s')}ds'\right)ds. \tag{D.21}$$

The value of $\overline{T}^{no}$ is obtained from the terminal condition $X^{no}(\overline{T}^{no}) = \overline{X} = \zeta\overline{T}^m$. We get:

$$\zeta\overline{T}^m = C_0\int_0^{\overline{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}d\tau - \lambda_0\zeta\int_0^{\overline{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}\left(\int_0^\tau e^{-\lambda_0 s}\sqrt{Z(s)}ds\right)d\tau. \tag{D.22}$$

Simplifying and using (D.20) to express C_0 yields (6.11).

Inserting the expression of C_0 from (D.20) into (D.19), we obtain the expression of $x^{no}(\tau)$ for $\tau \le \overline{T}^{no}$ given in (6.13). The expression $\tau \ge \overline{T}^{no}$ is straightforward.

Now, observing that $Z(\tau) \ge Z(\overline{T}^{no})$ for all $\tau \le \overline{T}^{no}$, we obtain the following majoration of the r.-h. side of (6.11) as

$$\overline{T}^m < e^{-\lambda_0\overline{T}^{no}}\left(\int_0^{\overline{T}^{no}} e^{\lambda_0\tau}d\tau\right)\sqrt{\frac{\lambda_0}{\lambda_1}} + \lambda_0\int_0^{\overline{T}^{no}} e^{-\lambda_0\tau}\left(\int_0^\tau e^{-\lambda_0 s}ds\right)d\tau$$

or, after simplifying,

$$\overline{T}^m < \overline{T}^{no} - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right)\frac{1 - e^{-\lambda_0\overline{T}^{no}}}{\lambda_0}.$$

From there and (4.20), it follows that $\overline{T}^{no} > \overline{T}^k$. *Q.E.D.*

APPENDIX E: EXTRA PROOFS

PROOF OF LEMMA B.1: Starting with the definition of $X^o(\tau; X)$ we get:

$$\frac{\partial X^o}{\partial \tau}(\tau; X) = \sigma^o(X^o(\tau; X)).$$

Differentiating with respect to X and using Schwartz' Lemma (for $X^o(\tau; X)$ twice continuously differentiable) yields

$$\frac{\partial}{\partial \tau} \log \left(\frac{\partial X^o}{\partial X}(\tau; X) \right) = \dot{\sigma}^o(X^o(\tau; X)).$$

Integrating and taking into account that $X^o(0; X) = X$ yields

$$\frac{\partial X^o}{\partial X}(\tau; X) = exp \left(\int_0^\tau \dot{\sigma}^o(X^o(s; X)) ds \right). \tag{E.1}$$

Using the stationarity of the feedback rule and differentiating with respect to t yields

$$\dot{\sigma}^o(X^o(\tau; X)) = \frac{\frac{\partial^2 X^o}{\partial \tau^2}(\tau; X)}{\frac{\partial X^o}{\partial \tau}(\tau; X)}. \tag{E.2}$$

Inserting into (E.1) and integrating yields

$$\frac{\partial X^o}{\partial X}(\tau; X) = exp \left(ln \left(\frac{\frac{\partial X^o}{\partial \tau}(\tau; X)}{\frac{\partial X^o}{\partial \tau}(0; X)} \right) \right)$$

and thus

$$\frac{\partial X^o}{\partial X}(\tau; X) = \frac{\sigma^o(X^o(\tau; X))}{\sigma^o(X^o(0; X))}.$$

Noticing that $X^o(0; X) = X$ yields (B.1). *Q.E.D.*

PROOF OF LEMMA B.2: Take $\tau > \varepsilon$, we have

$$\hat{X}(x, \varepsilon, \tau; X) = X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(\hat{X}(x, \varepsilon, s; X)) ds$$

Now observe that, for $s \geq \varepsilon$, we have

$$\hat{X}(x, \varepsilon, s; X) = X^o(s - \varepsilon, X + x\varepsilon).$$

Hence, we rewrite

$$\hat{X}(x, \varepsilon, \tau; X) = X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(X^o(s - \varepsilon, X + x\varepsilon)) ds. \tag{E.3}$$

Differentiating with respect to ε yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \int_0^\tau \dot{\sigma}^o(X^o(s; X)) \left(-\frac{\partial X^o}{\partial s}(s; X) + x \frac{\partial X^o}{\partial X}(s; X) \right) ds. \tag{E.4}$$

Inserting (B.1) into (E.4) yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1 \right) \int_0^\tau \dot{\sigma}^o(X^o(s; X)) \frac{\partial X^o}{\partial s}(s; X) ds.$$

Integrating the last term yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1 \right) \left(\sigma^o(X^o(\tau, X)) - \sigma^o(X) \right). \tag{E.5}$$

Simplifying further yields (B.2). *Q.E.D.*

Proof of Lemma B.3: Differentiating (5.3) with respect to τ yields

$$\frac{\partial Z}{\partial \tau}(\tau; X) = \dot{Z}^o(X^o(\tau; X))\sigma^o(X^o(\tau; X)). \tag{E.6}$$

Differentiating (5.3) with respect to X and using (B.1) now yields

$$\frac{\partial Z}{\partial X}(\tau; X) = \dot{Z}^o(X^o(\tau; X))\frac{\sigma^o(X^o(\tau; X))}{\sigma^o(X)}. \tag{E.7}$$

Gathering (E.6) and (E.7) yields (B.3). Using (B.3) and (5.3) and

$$Z(\tau; X) = (Z^o(X) - 1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau}\int_0^\tau F(X^o(s; X))e^{\Delta s}ds \quad \forall \tau \geq 0, X \geq 0, \tag{E.8}$$

finally yields (B.4).

Consider $Z_0(X) = 1 - F(X)$. Observe that $\dot{Z}_0(X) < 0$ when $f(X) > 0$. Observe also that $\dot{Z}^o(0) = 0 > \dot{Z}_0(0)$ when $\sigma^o(0) > 0$. Hence, $Z^o(X) > Z_0(X)$ in a starred-right neighborhood of 0. Suppose that $Z^o(X)$ crosses again $Z_0(X)$ for the first time at some $X_1 > 0$, the same reasoning as above shows that $\dot{Z}^o(X_1) = 0 > \dot{Z}_0(X_1)$ when $\sigma^o(X) > 0$ and thus $Z^o(X) < Z_0(X)$ in a starred-left neighborhood of X_1; a contradiction. Hence, $Z^o(X) \geq Z_0(X)$ for all X with equality at $X = 0$ only. From (B.3), $\dot{Z}^o(X) \leq 0$. *Q.E.D.*

Proof of Lemma B.4: If $\mathcal{V}^o(X)$ is continuously differentiable, $\hat{\mathcal{V}}(x, \varepsilon; X)$ is itself continuously differentiable in ε, and a first-order Taylor expansion in ε yields

$$\hat{\mathcal{V}}(x, \varepsilon; X) = \mathcal{V}^o(X) + \varepsilon\frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X) + o(\varepsilon). \tag{E.9}$$

Hence, (5.12) amounts to (B.5). Conjectures being correct at equilibrium, (B.6) also holds. *Q.E.D.*

A Verification Theorem. Proposition E.1 below shows that the conditions given Proposition 1 to characterize the extended value function by means of an Hamilton-Bellman-Jacobi equation together with boundary conditions are in fact sufficient. We follow Ekeland and Turnbull (1983, Theorem 1, p. 6) to derive a *Verification Theorem*.

Proposition E.1 *Assume first that there exists a continuously differentiable function $\mathcal{W}_0(X, Z)$ which satisfies:*

(E.10)

$$\lambda_0\mathcal{W}_0(X, Z) \geq Z(t; X, Z)u(x) + x\frac{\partial \mathcal{W}_0}{\partial X}(X, Z) + \Delta(1 - F(X) - Z(t; X, Z))\frac{\partial \mathcal{W}_0}{\partial Z}(X, Z) \quad \forall (x, X, Z);$$

and, second, that there exists an action profile X and a path $\overline{X}(t) = \int_0^t \overline{X}(\tau)d\tau$, $Z_0(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(\overline{X}(\tau))e^{\Delta\tau}d\tau$ such that

$$\lambda_0\mathcal{W}_0(\overline{X}(t), Z_0(t)) = Z_0(t)u(\overline{X}(t)) + \overline{X}(t)\frac{\partial \mathcal{W}_0}{\partial X}(\overline{X}(t), Z_0(t)) + \Delta(1 - F(\overline{X}(t)) - Z_0(t))\frac{\partial \mathcal{W}_0}{\partial Z}(\overline{X}(t), Z_0(t)) \quad \forall t \geq 0. \tag{E.11}$$

Then X is an optimal action profile with its associated path $(\overline{X}(t), Z_0(t))$.

Proof of Proposition E.1: Suppose that a function $\mathcal{W}^e(X, Z)$ that satisfies conditions in Proposition A.2 is continuously differentiable. It is our candidate for the function $\mathcal{W}_0(X, Z)$ in the statement of Proposition E.1. By definition (A.24), we have

$$\lambda_0\mathcal{W}^e(X, Z) = Zu(\sigma^e(X, Z)) + \sigma^e(X, Z)\frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \Delta(1 - F(X) - Z)\frac{\partial \mathcal{W}^e}{\partial Z}(X, Z), \quad \forall (X, Z)$$

and thus

$$\lambda_0\mathcal{W}^e(X,Z) \geq Zu(x) + x\frac{\partial \mathcal{W}^e}{\partial X}(X,Z) + \Delta(1-F(X)-Z)\frac{\partial \mathcal{W}^e}{\partial Z}(X,Z), \quad \forall (x,X,Z) \tag{E.12}$$

where the inequality comes from the fact that $\sigma^e(X,Z)$ maximizes the r.-h.s..

To get (E.11), we use again (A.24) but now applied to the path $(x^e(t), X^e(t), Z^e(t))$ where $X^e(t)$ is such that $\dot{X}^e(t) = x^e(t) = \sigma^e(X^e(t), Z^e(t))$ with $X^e(0) = 0$ and $Z^e(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(x^e(\tau))e^{\Delta\tau}d\tau$.

Define now a value function $\widetilde{\mathcal{W}^e}(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}^e(X,Z)$. By (E.12), we get

(E.13)

$$0 \geq \frac{\partial \widetilde{\mathcal{W}^e}}{\partial t}(X,Z,t) + x\frac{\partial \widetilde{\mathcal{W}^e}}{\partial X}(X,Z,t) + \Delta(1-F(X)-Z)\frac{\partial \widetilde{\mathcal{W}^e}}{\partial Z}(X,Z,t) + e^{-\lambda_0 t}Zu(x) \quad \forall (x,X,Z).$$

Using $X^e(t) = \sigma^e(X^e(t), Z^e(t))$, $Z^e(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(x^e(\tau))e^{\Delta\tau}d\tau$ and (E.11), we get

$$\begin{aligned} 0 = {} & \frac{\partial \widetilde{\mathcal{W}^e}}{\partial t}(X^e(t), Z^e(t), t) + x^e(t)\frac{\partial \widetilde{\mathcal{W}^e}}{\partial X}(X^e(t), Z^e(t), t) \\ & + \Delta(1 - F(X^e(t)) - Z^e(t))\frac{\partial \widetilde{\mathcal{W}^e}}{\partial Z}(X^e(t), Z^e(t), t) + e^{-\lambda_0 t}Z^e(t)u(X^e(t)) \quad \forall t \geq 0. \end{aligned} \tag{E.14}$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t) = \int_0^t x(\tau)d\tau$ and $Z(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X(\tau))e^{\Delta\tau}d\tau$. Eventually, this path crosses the upper bound $\overline{X}$ at some $\overline{T}^e$. Let us fix an arbitrary $t > 0$. Integrating (E.13) along the path $(x(\tau), X(\tau), Z(\tau))$, we compute

$$\begin{aligned} 0 \geq \int_0^t \Bigg(& \frac{\partial \widetilde{\mathcal{W}^e}}{\partial \tau}(X(\tau), Z(\tau), \tau) + x(t)\frac{\partial \widetilde{\mathcal{W}^e}}{\partial X}(X(\tau), Z(\tau), \tau) \\ & + \Delta(1 - F(X(\tau)) - Z(\tau))\frac{\partial \widetilde{\mathcal{W}^e}}{\partial Z}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0\tau}Z(\tau)u(x(\tau)) \Bigg) d\tau \end{aligned}$$

or

$$0 \geq \int_0^t \left(\frac{d\widetilde{\mathcal{W}^e}}{d\tau}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0\tau}Z(\tau)u(x(\tau)) \right) d\tau \quad \forall t \geq 0.$$

Integrating the first term on the r.-h.s., we thus get

$$\widetilde{\mathcal{W}^e}(0,0,0) \geq \widetilde{\mathcal{W}^e}(X(t), Z(t), t) + \int_0^t e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau \quad \forall \tau \geq 0.$$

Because $\widetilde{\mathcal{W}^e}(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}^e(X,Z) \geq 0$ for all (X,Z,t), we obtain:

$$\mathcal{W}^e(0,0) \geq e^{-\lambda_0 t}\mathcal{W}^e(X(t), Z(t)) + \int_0^t e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau \quad \forall \tau \geq 0.$$

Because of the boundary conditions (A.29), $e^{-\lambda_0 t}\mathcal{W}^e(X(t), Z(t))$ converges towards zero as $t \to +\infty$ for any feasible path. Moreover, for any such feasible path $\int_0^{+\infty} e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau$ exists. Henceforth, we get:

$$\mathcal{W}^e(0,0) \geq \sup_{\mathbf{x}} \int_0^{+\infty} e^{-\lambda_0\tau}Z(\tau)u(x(\tau))dt$$

which shows that $(x^e(\tau), X^e(\tau), Z^e(\tau))$ is indeed an optimal path. *Q.E.D.*