MATHEMATICAL MODELS OF DYNAMIC EQUILIBRIA FOR LARGE POPULATIONS, AND THEIR APPLICATIONS

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WHAT? WHY?

- decentralized / centralized intelligence
- mathematical models for situations involving many agents/players

I. INTRODUCTION

- New class of models for the average (Mean Field) behavior of "small" agents (Games) started in the early 2000's by J-M. Lasry and P-L. Lions.
- Requires new mathematical theories.
- Numerous applications: economics, finance, social networks, crowd motions, telecommunications, Meaningful Data and ML. . .
- Independent introduction of a particular class of MFG models by M. Huang, P.E. Caines and R.P. Malhamé in 2006.
- Previous related works in Economics: anonymous games, Krusell-Smith. . .
- A research community in expansion: mathematics, economics, finance, telecommunications, energy. . .

- In 2018, two books; most of the existing mathematical material to be found in the Collège de France videotapes ($\geqslant \times$ 18h) that can be downloaded. . . !
- Combination of Mean Field theories (classical in Physics and Mechanics) and the notion of Nash equilibria in Games theory.
- Nash equilibria for continua of "small" players: a single heterogeneous group of players (adaptations to several groups. . .).
- Each generic player is "rational" i.e. tries to optimize (control) a criterion that depends on the others (the whole group) and the optimal decision affects the behavior of the group (however, this interpretation is limited to some particular situations. . .).
- Huge class of models: agents $\rightarrow$ particles, no dep. on the group are two extreme particular cases.

AN EXAMPLE OF APPLICATIONS: CROWD MOTIONS

II. A REALLY SIMPLE EXAMPLE

- Simple example, not new but gives an idea of the general class of models (other "simple" exs later on): where do we put our towels on the beach?
- E metric space, N players ($1 \leqslant i \leqslant N$) choose a position $x_i \in E$ according to a criterion $F_i(X)$ where $X = (x_1, \dots, x_N) \in E^N$.
- Nash equilibrium: $\bar{X} = (\bar{x}_1, \dots, \bar{x}_N)$ if for all $1 \leqslant i \leqslant N$ $\bar{x}_i$ min over E of $F_i(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots \bar{x}_N)$.
- Usual difficulties with the notion
- $N \to \infty$? simpler ?
- Indistinguishable players:

$$F_i(X) = F(x_i, \{x_j\}_{j \neq i}), F \text{ sym . in } (x_j)_{j \neq i}$$

- Part of the mathematical theories is about $N \to \infty$:

$$F_i = F(x, m) \quad x \in E \ , \quad m \in \mathcal{P}(E)$$

$$\text{where } \ x = x_i \ , \ m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$$

- "Thm": Nash equilibria converge, as $N \to \infty$, to solutions of

$$\text{(MFG)} \qquad \forall x \in \ \text{Supp}\, m, F(x, m) = \inf_{y \in E} \ F(y, m)$$

- Facts: i) general existence and stability results

  ii) uniqueness if $(m \to F(\bullet, m))$ monotone

  iii) If $F = \Phi'(m)$, then $(\min_{\mathcal{P}(E)} \Phi)$ yields one solution of MFG.

Example: $E = \mathbb{R}^d, F_i(X) = f(x_i) + g\left(\frac{\#\{j/|x_i - x_j| < \varepsilon\}}{(N-1)|B_\varepsilon|}\right)$

$g \uparrow$ aversion crowds, $g \downarrow$ like crowds

$$F(x, m) = f(x) + g(m * 1_{B_\varepsilon}(x)(|B_\varepsilon|^{-1})$$

$$\varepsilon \to 0 \qquad F(x, m) = f(x) + g(m(x))$$

$$\text{(MFG)} \qquad \mathrm{supp}\ m \subset \text{Arg min}\left(f(x) + g(m(x))\right)$$

– $g \uparrow$ uniqueness, $g \downarrow$ non uniqueness

$$\min\left\{\int fm + \int G(m)/m \in \mathcal{P}(E)\right\}, \ G = \int_0^Z f(s)ds$$

– explicit solution if $g \uparrow$: $m = g^{-1}(\lambda - f), \lambda \in \mathbb{R}\ s.t. \int m = 1$

III. GENERAL STRUCTURE

- MFG dynamical equilibria lead to equations

$$\frac{\partial U}{\partial t} + D + T + N = 0 \quad t \in (0, T)$$

where the unknown function U ("value function")

$$U(x, m, t) \; \in \mathbb{R}$$

(this is an example: finite horizon, one crowd? ...)

- D : decision block, for instance, optimal control decision

$$D = H(x, m, \partial_x U) \quad \text{(Bellman)}$$

optimal control $\alpha^*(x, m, \partial_x u)$

- T : transport block, for instance, the state of the population is modified only by the individual decisions

$$T = \langle \partial_m U, -\partial_x \dot{\{} B(x, m; \alpha^*) m \} \rangle$$

- N : random effects (idiosynchratic noise variance a, common noise variance b)

$$N = -\frac{a+b}{2}\,\partial_x^2 U + \langle \partial_m U, -\frac{a+b}{2}\,\partial_x^2 m) \rangle + b \langle \partial_x \dot{\partial}_m U, \partial_x m \rangle$$

- A VERY PARTICULAR CASE: dynamical problem, horizon T, continuous time and space, Brownian noises (both indep. and common), no intertemporal preference rate, control on drifts (Hamiltonian H), criterion dep. only on m

- $U(x, m, t)$ $(x \in \mathbb{R}^d, m \in \mathcal{P}(\mathbb{R}^d)$ or $\mathcal{M}_+(\mathbb{R}^d), t \in [0, T]$ and $H(x, p, m)$ (convex in $p \in \mathbb{R}^d$)

- MFG master equation (∞d equation!)

$$\begin{cases} \frac{\partial U}{\partial t} - (\nu+\alpha)\Delta_x U + H(x, \nabla_x U, m) + \\ + \langle \frac{\partial U}{\partial m}, -(\nu+\alpha)\Delta m + \operatorname{div}\,(\frac{\partial H}{\partial p} m)\rangle + \\ -\alpha \frac{\partial U}{\partial m^2}(\nabla m, \nabla m) + 2\alpha \langle \frac{\partial}{\partial m} \nabla_x U, \nabla m\rangle = 0 \end{cases}$$

and $U\mid_{t=0} = U_0(x, m)$ (final cost)

- ν amount of ind. rand. , α amount of common rand.

TWO PARTICULAR CASES

- ∞ d problem in general but reductions to finite d in two cases

1. Indep. noises ($\alpha = 0$), no common noise

int. along caract. in m yields

$$(\text{MFGi}) \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u, m) = 0 \\ u\,|_{t=0} = U_0(x, m(0)),\, m\,|_{t=T} = \bar{m} \\ \frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(\frac{\partial H}{\partial p} m\right) = 0 \end{cases}$$

where $\bar{m}$ is given

FORWARD — BACKWARD system !

2. Finite state space ($i \leqslant i \leqslant k$): graphs...

(MFGf) $\frac{\partial U}{\partial t} + (F(x, U) \,.\, \nabla)\, U = G(x, U), U\,|_{t=0} = U_0$

(no common noise here to simplify ...)

$$x \in \mathbb{R}^k,\ U \to \mathbb{R}^k,\ F \text{ and } G : \mathbb{R}^{2k} \to \mathbb{R}^k$$

non-conservative hyperbolic system

Example: If $F = F(U) = H'(U), G \equiv 0$

and if $U_0 = \nabla\varphi_0$ $(\varphi_0 \to \mathbb{R})$ then

– solve HJ

$$\frac{\partial \varphi}{\partial t} + H(\nabla\varphi) = 0\ ,\ \varphi\,|_{t=0} = \varphi_0$$

– take $U = \nabla\varphi$, "U solves" (MFGf) in this case

FUNCTIONS OF MANY VARIABLES

- From $u(x_1, \ldots, x_N)$ symmetric in $(x_1, \ldots, x_N)$ to $u(m)$ $m \in \mathcal{P}$ (think $m = \frac{1}{N}\sum_i \delta_{x_i}$ (empirical measures) or equivalently $u(X)$ X random variable whose law is m (think $X = x_i$ with probability $1/N$)

- mathematical theory to exploit this idea with applications in particular to the Master Equation.

IV. OVERVIEW AND PERSPECTIVES

Lots of questions, partial results exist, many open problems

- Existence/regularity:
  - (MFGi) "simple" if H "smooth" in m (or if H almost linear ...), OK if monotone
  - (MFGf) OK if (G, F) mon. on $\mathbb{R}^{2k}$ or small time
- Uniqueness: OK if "monotone" or T small ...
- Non existence, non uniqueness, non regularity (!)
- Qualitative properties, stationary states and stability, comparison, cycles ...
- $N \to \infty$ (see above)
- Numerical methods (currently, 3 "general" methods and some particular cases)
- Variants: other noises, several populations, more couplings ...
- applications (MFG Labs ...)

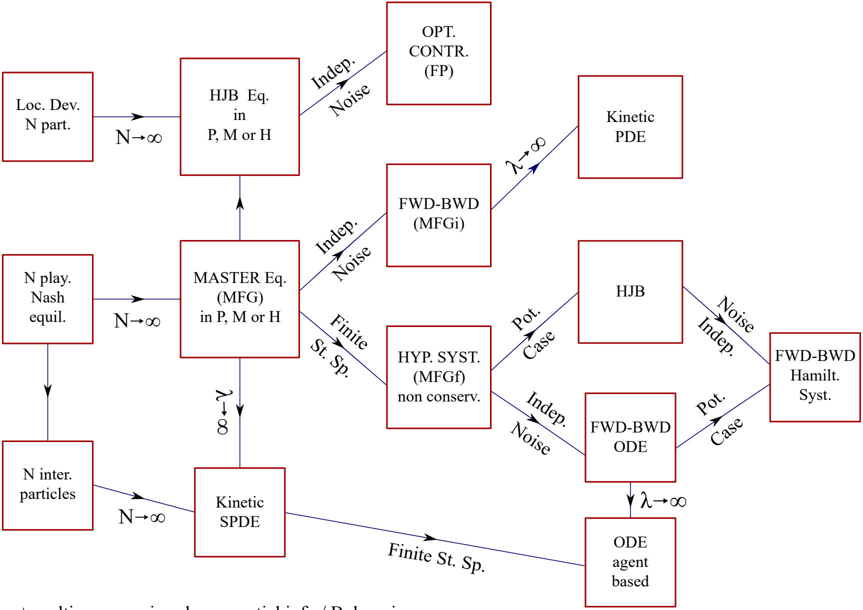
NEW • optimal stopping, impulsive controls

NEW • intertemporal preference rates ($+\lambda \to \infty$): agent based models, kinetic models...

NEW • MFG with a major player

NEW • MFG with Bayesian learning/partial information

• ? Beyond MFG ? (fluctuations, LD, transitions)



+ multi. pop., major player, partial info / B. learning
∞D state sp., opt. stopping / impulsive ...

TWO S. EXAMPLES

- at which time will the meeting start?
- the mexican wave

V. MEANINGFUL DATA

- MFG Labs
- Practical expertise and models mainly for "big" data involving "people"
- New models that include classical clustering models in M.L. (K-mean, EM ...), then algorithms
- No need for euclidean structures or for "a priori" distances
- Models for Deep Learning

- Why “PEOPLE”: STRATEGY!

  Ex. 1: Taxis

  Ex. 2: Movies and Fb

People that are “close” will say they like movies that are “close”

→ consistency distance - like on items/people

VI. RECENT THEORETICAL DEVELOPMENTS

1. WEAK FORM OF MASTER EQS (Ch. Bertucci)
   - requires "less" regularity
   - using monotonicity structure
2. EXTENDED MFG (P.E. Souganidis - PL^2)
   - "decouples" $\frac{\partial H}{\partial p}$ term in m-dynamics and the H term for the value equation
   - Example:
     $-\frac{\partial u}{\partial t} + A(\nabla u, m) = 0, \frac{\partial u}{\partial t} + \mathrm{div}(B(\nabla u, m)) = 0$
   - unifies MFG and Optimal Control/Transport
   - stable by homogenization limits
3. FINITE STATE SPACE TO CONTINUA (. . .)
   - asymptotic limit
   - OK under monotonicity conditions

4. PARTIAL OBSERVATION/INFORMATION

4.1 PARTIAL INFORMATION AND BAYESIAN LEARNING (PL^2, CdF)

- agent does not "know" his controlled drift
- 1 agent: Stochastic Control with Partial Information $\to$ leads to the optimal control of Zakaï 's equation (PL^2 a long time ago...)
- agents share the same belief on the others
- MASTER EQUATION (Proba on Proba!)
- Reduction in the "Gaussian" case

4.2 PARTIAL OBSERVATION (Ch. Bertucci)

- agent knows his state but does not know the state of the crowd
- belief on the state of the crowd (Proba on Proba 2)
- MASTER EQUATION
- well-posedness under monotonicity conditions

5. MAJOR PLAYER/CROWD WITH STRATEGIC ADVANTAGES
(J.M. Lasry-PL[2], Ch. Bertucci, J.M. Lasry-PL[2])
6. A "CATALOGUE" OF SOLUTIONS (B. Seeger-PL[2])
Ex: FINITE STATE SPACE, functions nonincreasing in all variables
    - $\exists$ maximal, minimal solutions (all solutions between...)
    - Different regularisations pick various solutions (maximal, minimal, others...)
    - Examples with a complete description of all solutions
7. NUMERICAL SIMULATION OF MASTER EQS VIA NEURAL NETS
(Y. Achdou, L. Bertucci, J-M. Lasry, PL[2])
    - ∞D nonlinear equation !
    - why it might be possible/why it is possible
    - Ex. Krussel-Smith

8. RANDOM MATRICES (non commutative spaces):

- asymptotic integro-differential Vlasov-Mc Keen equations (Ch. Bertucci, M. Debbah, J-M. Lasry, PL^2)
- optimal control of systems governed by large random matrices

  $\rightarrow$ optimal control of above eqs $\rightarrow$ MFG
- MFG for intelligent systems governed by large random matrices (yet to be explored)
- RK: "Similar problems" for preferential attachments networks/graphs

VII. RECENT APPLICATIONS

1. MOBILE NETWORKS (Bertucci, Debbah, Paschos, Lasry, Vassilaras, PL^2)
   CROWD OF DEVICES CONNECTED TO AN ANTENNA (5G): MINIMIZE ENERGY USAGE WHILE ACHIEVING QOS REQUIREMENTS (IEEE, 15$^{\text{th}}$ ISWGS) and more in preparation...
2. BITCOIN MINING (Bertucci^2, Lasry, PL^2)
   COMPETITION BETWEEN MINERS in a POW based BLOCKCHAIN
   unique equilibrium (MFG), total compact power..., and other "proofs", lightning networks...
3. MACHINE LEARNING (in collaboration with F. BACH group at INRIA)
   mathematical models for machine learning (my seminar at CdF 11/09/18 "deep learning")

- many existing works with mean-field limits, not games
- various issues: number of neurons going to infinity, number of layers going to infinity, game interpretation a posteriori

4. OIL PRODUCTION (Achdou, Bertucci, Lasry, Rostand, Scheinkman, PL^2)
   - major agent (cartel), competitive producers and arbitrages (via storage)
   - MFG models (EDMOND 1, 2)
   - calibration on historical data
   - agreement on production shares ...
   - new predictions: the "cliff", negative prices in some situations
   - article in FT (hugely read) on March 23, 2020 saying that our game theoretic work explains the current oil crisis (falling off "the cliff")
   - in April 2020, negative prices were observed (in particular for WTI oil futures)