Decision-Making When Things Are Only a Matter of Time

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Abstract

This article gives a comprehensive treatment of preferences regarding time risk—the risk of something happening sooner or later—within the expected discounted utility model. We characterize the signs of the discount function's derivatives of all orders and show how these signs are decisive for time risk preferences. We introduce the notions of *prudent* and *temperate* discounting, and illustrate their importance for economic behavior. Several applications in which an important event is "only a matter of time" are presented. The seminal net present value rule to evaluate investment opportunities is generalized to take into account the timing uncertainty of cash flows.

Keywords: risk preferences, time preferences, time risk preferences, prudence, precautionary behavior, investment

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a matter (also question) of time: "said with reference to an event or circumstance that is thought certain to come about, or to resolve itself in a particular way, sooner or later."—*Oxford English Dictionary*[1]

1 Introduction

When researchers speak of decision-making under risk, they usually refer to the risk of *what* may happen. The outcome at risk may be monetary or it may concern one's health; it may affect current or future well-being; and it may concern oneself and/or others. This article contrasts these well-studied preferences toward *outcome risk* with preferences toward a fundamentally different type of risk that is about *when* something may happen: *time risk.* While most articles on risk assume perfect knowledge about when outcomes materialize and that uncertainty is restricted to how good or bad the outcome will be, this article studies the dual case in which the outcome is known but the time of its occurrence is not. Everyday language refers to such outcomes as being "only a matter of time." This article studies preferences toward the risk that this time may be shorter or longer or, equivalently, that the outcome that is only a matter of time occurs sooner or later.

The contribution of the article is twofold. First, we give a comprehensive account of time risk preferences within the expected discounted utility (EDU) model, by identifying and characterizing the properties of the discount function that are decisive for choices over simple time risks (time lotteries). Second, we consider a number of applications of these results, which illustrate the importance of time risk preferences for behavior in situations in which a known outcome is only a matter of time.

The key idea behind the first contribution—the identification and characterization of properties of the discount function that are decisive for decision-making under time risk—is to exploit an analogy between time risk preferences and outcome risk preferences. Choices over time risks are determined by a comparison of their "expected discounts." This comparison is mathematically similar to that of "expected utilities," which determines choices over outcome risks. Section 2 formalizes the exact analogy between choices over time risks and choices over outcome risks within a simple (mathematically trivial) result referred to as the *time-outcome risk duality.* We use it to convert known results about utility functions to obtain yet unknown results about discount functions. For example, using the time-outcome risk duality, one can translate Pratt's (1964) seminal result on comparative risk aversion to obtain a characterization of the Arrow-Pratt measure

[1] "time, n., int., and conj. Phrase P1 d." *Oxford English Dictionary,* OED Online, Third Edition (March 2012), Oxford University Press. Last retrieved: 26 July 2019. The hyperlink requires access to OED Online.

of a discount function.

In general, the translation of results on outcome risk preferences into results on time risk preferences is not straightforward. The curvature of utility functions—which determines outcome risk preferences in expected utility (EU)—typically differs from that of discount functions. While the former are typically increasing and concave, and maybe unbounded, the latter are typically decreasing, convex, and bounded. These properties (and others) matter for risk preferences, and thus the translation of results using the time-outcome risk duality often changes their behavioral implications. Moreover, for time risk preferences, it is decisive whether the outcome that is only a matter of time is desirable or not. Depending on this desirability, for example, convex discounting may imply a risk-seeking or a risk-averse attitude toward time risks.

The necessary tool to advance the analysis of time risk preferences beyond the basic notion of risk aversion is the *discount premium,* which we define in analogy to the utility premium of Friedman and Savage (1948). The discount premium measures the change in the (expected) discount factor due to making the outcome's time of receipt risky. As the higher-order (outcome) risk preferences of prudence ($u''' \geq 0$ in EU, Kimball 1990) and temperance ($u'''' \leq 0$, Kimball 1993) complement risk aversion in important ways, we define and investigate the *higher-order time risk preferences* of *prudent discounting* and *temperate discounting.* The definitions are based on preferences over simple risk apportionment lottery pairs (Eeckhoudt and Schlesinger 2006) and can be characterized through conditions on the signs of the discount functions' (higher-order) derivatives. We observe that the commonly used discount functions are decreasing with derivatives that alternate in sign, which, in the case of desirable outcomes, imply risk-seeking ($\rho'' \geq 0$), prudent ($\rho''' \leq 0$), and *in*temperate ($\rho'''' \geq 0$) discounting. In analogy to Caballé and Pomansky's (1996) mixed (outcome) risk preferences, we call the time risk preferences implied by these discount functions *anti-mixed.* We characterize anti-mixed time preferences through a preference for properly apportioning fixed and risky delays across possible states of nature, in a consistent manner, similarly to the utility case studied in Eeckhoudt et al. (2009) and Crainich et al. (2013).

Per the second contribution of the article, we illustrate the relevance of time risk preferences as such (and, in particular, that of the new properties of prudence, temperance, anti-mixedness, etc.) through a number of applications. We show that time risk preferences are crucial for behavior in situations in which an important event (e.g., a large payment, a change in one's health, or any other important personal event) is only a matter of time. Discount prudence is related to precautionary behavior toward time risk, and both discount prudence and discount temperance are shown to be important for investment decisions that involve an uncertain time of repayment. We also give a new characterization of exponential discounting, through time risk preferences. Moreover, we provide a generalization of the seminal net present value (NPV) rule that takes into account that, in reality, future cash flows and other outcomes may occur sooner or later. Further applications, generalizations, and limitations of our results are discussed at the end of the article.

This article contributes to the literature on risk and time preferences by focusing on preferences toward time risks. We are concerned with "the study of time as the duration that a person has to wait until she receives an outcome in intertemporal choice (discounting)" (Abdellaoui and Kemel 2014) [p. 1844], and consider the case in which this receipt time is risky. This study of time is not to be confused with that of time as a resource.[2] Let us also clarify how this article relates to the topic of sequential resolution of uncertainty (Kreps and Porteus 1978). The uncertainty whose sequential resolution is studied has traditionally concerned outcome uncertainty (e.g., "When will I get to know how large my year-end bonus will be?"). But, one may likewise study the sequential resolution of time risk (e.g., "When will I get to know when my $10,000 bonus will be paid?"). In line with the few existing articles on time risks (discussed next), we focus on the case in which the resolution of risk is immediate.

On the theory side, the first analysis of time risk appears in Chesson and Viscusi (2003), who show that exponential discounting implies time risk-seeking toward desirable outcomes. For that case, Onay and Öncüler (2007) prove that time risk-seeking, in the sense of every risk being preferred over its expected value, holds for any convex discount function. The authors further analyze time risk aversion in a model with probability weighting and consider the case of undesirable outcomes. DeJarnette et al. (2018) study time risk aversion and show that, within EDU, it is incompatible with a new property called stochastic impatience. This incompatibility remains in much more general models than EDU. DeJarnette et al. (2018) further provide an axiomatization of EDU preferences over time risks when discounting is exponential. Chesson and Viscusi (2003), Onay and Öncüler (2007), and DeJarnette et al. (2018) all provide empirical (survey or experimental) evidence on time risk aversion. Ebert (2018) experimentally investigates prudent and temperate discounting, complementing analogous evidence on outcome prudence and outcome temperance (e.g., Deck and Schlesinger 2010, Ebert and Wiesen 2011, and Maier and Rüger 2011). The current article, by going beyond the property of time risk-seeking/the convexity of the discount function, presents a more

[2]Leclerc et al. (1995), Abdellaoui and Kemel (2014), Festjens et al. (2015), and Noussair and Stoop (2015), among others, conduct experiments in which the outcome of subjects' decisions is waiting shorter or longer in a room without the possibility to do anything. In that case, time is an *outcome* and choices over risks in waiting time identify the *utility of time.* Choices over time risks instead identify discounting (of a fixed amount of utility). Recently, Abdellaoui et al. (2018) have conducted the first experiment that involves both types of "time" and elicit preferences toward the delay of outcome time (i.e., waiting time).

comprehensive treatment of time risk preferences within EDU. Moreover, it is the first to study behavioral implications of time risk preferences alongside various economic models.

2 The time-outcome risk duality

In this short section, we introduce notation and clarify the analogy between time risk and outcome risk preferences. That is, we explain the "channel" through which known results about utility functions can be given a correspondence in the realm of discount functions. Throughout the article consider a strictly increasing *utility function* $u : \mathcal{X} \longrightarrow (-\infty, \infty)$ that maps outcomes x (e.g., monetary prizes, health states, etc.) taken from a non-empty ordered set $\mathcal{X}$ to utility levels $u(x)$. For an element $x_0 \in \mathcal{X}$, $u(x_0) = 0$. Let $\rho : [0, \infty) \longrightarrow (0, 1]$ with $\rho(0) = 1$ denote a smooth and strictly decreasing *discount function,* mapping dates t to discount factors $\rho(t)$. Let Δ be the set of simple lotteries on $[0, \infty) \times \mathcal{X}$. Consider preferences $\succeq$ over elements $(\tilde{\tau}, \tilde{x}) \in \Delta$ that are represented by a functional U given by

$$U(\tilde{\tau}, \tilde{x}) := \mathbb{E}[\rho(\tilde{\tau})u(\tilde{x})], \tag{1}$$

referred to as the decision-maker's (DM's) *expected discounted utility (EDU).* For most of the article, we are concerned with the evaluation of more specific elements of Δ. Let $(\tau, x) \in \Delta$ denote the degenerate risk that yields the certain outcome x at the certain time τ, referred to as a *delayed outcome.* It is evaluated as $U(\tau, x) = \rho(\tau)u(x)$. Taking this riskless case as a benchmark, note that a large body of literature studies the generalization in which the outcome that is paid at the certain time τ is risky. Indicating the riskiness of an outcome by $\tilde{x}$, the resulting *(delay-certain) outcome risk* $(\tau, \tilde{x}) \in \Delta$ is evaluated as

$$U(\tau, \tilde{x}) = \mathbb{E}[\rho(\tau)u(\tilde{x})]. \tag{2}$$

The DM prefers the delay-certain outcome risk $(\tau, \tilde{x}_1)$ over the delay-certain outcome risk $(\tau, \tilde{x}_2)$—note that they are dated to the same time τ—if and only if

$$\begin{aligned} U(\tau, \tilde{x}_1) \geq U(\tau, \tilde{x}_2) &\Longleftrightarrow \rho(\tau)\mathbb{E}[u(\tilde{x}_1)] \geq \rho(\tau)\mathbb{E}[u(\tilde{x}_2)] \\ &\Longleftrightarrow \mathbb{E}[u(\tilde{x}_1)] \geq \mathbb{E}[u(\tilde{x}_2)]. \end{aligned} \tag{3}$$

We see that the delay τ and the discount function $\rho(\cdot)$ drop out of the inequality. It follows that, in EDU, preferences over risky outcomes dated to the same certain time are determined by the curvature of utility alone.

Now we go back to the "delay-certain, certain outcome" (τ, x) and consider the mathematically symmetric generalization of making τ risky, thus denoting it by $\tilde{\tau}$, while letting the outcome x remain certain. The resulting object $(\tilde{\tau}, x) \in \Delta$ is an *(outcome-certain) time risk* or an *(outcome-certain) delay risk* for which the amount to be received is certain but the time at which it is received is not. In other words, the receipt of x is only a matter of time, and it is evaluated as follows:

$$U(\tilde{\tau}, x) = \mathbb{E}[\rho(\tilde{\tau})u(x)]. \tag{4}$$

The DM prefers the outcome-certain delay risk $(\tilde{\tau}_1, x)$ over the outcome-certain delay risk $(\tilde{\tau}_2, x)$—note that they yield the same outcome x—if and only if

$$U(\tilde{\tau}_1, x) \geq U(\tilde{\tau}_2, x) \Longleftrightarrow u(x)\mathbb{E}[\rho(\tilde{\tau}_1)] \geq u(x)\mathbb{E}[\rho(\tilde{\tau}_2)]$$

and, therefore:

Observation 1. (Time-outcome risk duality). *Consider EDU preferences $\succeq$ over delay-certain outcome risks $(\tau, \tilde{x}_1)$ and $(\tau, \tilde{x}_2)$ and certain-outcome delay risks $(\tilde{\tau}_1, x)$ and $(\tilde{\tau}_2, x)$, respectively. We have*

$$(\tau, \tilde{x}_1) \succeq (\tau, \tilde{x}_2) \Longleftrightarrow \mathbb{E}[u(\tilde{x}_1)] \geq \mathbb{E}[u(\tilde{x}_2)] \text{ and} \tag{i}$$

$$(\tilde{\tau}_1, x) \succeq (\tilde{\tau}_2, x) \Longleftrightarrow \begin{cases} \mathbb{E}[\rho(\tilde{\tau}_1)] \geq \mathbb{E}[\rho(\tilde{\tau}_2)] & \text{if } u(x) \geq 0 \\ \mathbb{E}[\rho(\tilde{\tau}_1)] \leq \mathbb{E}[\rho(\tilde{\tau}_2)] & \text{if } u(x) < 0. \end{cases} \tag{ii}$$

Observation 1 illustrates that choices over *bivariate* (certain-outcome) time risks are determined similarly to choices over *univariate, atemporal* outcome risks. The former are determined by comparing expected discounts, while the latter are determined by comparing expected utilities. Due to the mathematical similarity of these comparisons, we can utilize our existing knowledge about outcome risk preferences to get a better understanding of time risk preferences.

There are also, however, differences. Due to the different curvatures of discount and utility functions, choices over time risks may be very different from choices over outcome risks. An obvious difference is the need for a case distinction between desirable ($u(x) > 0$) and undesirable ($u(x) < 0$) outcomes in equivalence (ii), which is unique to the study of time (risk) preferences. This difference is a consequence of the fact that negative utility can have important meaning, while discount factors are safely assumed to be positive. We first study the case $u(x) > 0$ and turn to the case $u(x) < 0$ in Section 5.

3 A behavioral characterization of time risk preferences when outcomes are desirable

This section characterizes risk-taking behavior over (outcome-certain) time risks $(\tilde{\tau}, x) \in \Delta$ when $u(x) > 0$. We define risk-averse, prudent, and temperate discounting in analogy to Eeckhoudt and Schlesinger (2006) through preferences over simple and intuitive "time risk apportionment" lotteries, and show that these behavioral definitions are equivalent to signing various derivatives of the discount function. We show that the time risk preferences of commonly used "anti-mixed" discount functions can be given a simple intuition in terms of apportioning fixed and risky delays across states of nature.

3.1 Decreasing discount functions

Consider a pair of delayed outcomes (i.e., degenerate outcome-certain time risks) $(\tilde{\tau}_A^{(1)}, x) := (s, x)$ and $(\tilde{\tau}_B^{(1)}, x) := (l, x)$ that, at respective times $0 \leq s < l < \infty$, yield the same certain outcome x. The superscript "$^{(1)}$" is used to indicate that these delayed outcomes can be used to characterize first-order time "risk" preferences, that is, the first derivative of the discount function. Later subsections will define lotteries with higher superscripts since they will be shown to characterize higher-order time risk preferences, that is, the signs of higher-order derivatives of the discount function. As both first-order lotteries are degenerate, they may be depicted as

$$(\tilde{\tau}_A^{(1)}, x) \xrightarrow{\quad 1 \quad} (s, x) \qquad\qquad (\tilde{\tau}_B^{(1)}, x) \xrightarrow{\quad 1 \quad} (l, x).$$

Indeed, under the smoothness of the discount function assumed throughout, it follows from Observation 1 that $\rho'(\tau) \leq 0$ for all $\tau \geq 0$, if and only if $(\tilde{\tau}_A^{(1)}, x) \succeq (\tilde{\tau}_B^{(1)}, x)$ for all s, l.

3.2 Risk-averse discounting

Onay and Öncüler (2007) and DeJarnette et al. (2018) have established the important result that the convexity of the discount function implies weak risk-seeking for outcome-certain time risks. Note that, with Observation 1 in mind, this result also follows from Rothschild and Stiglitz (1970) [Theorem 2] for general mean-preserving spreads. Indeed, Rothschild and Stiglitz (1970) [footnote 3] note that "It might be argued that we should limit our discussion to increasing concave functions. Imposing this restriction would gain nothing and would destroy the symmetry of the results." Our application of their result to decreasing and convex discount functions makes practical use of this symmetry.

Next, we explain the sense in which a convex discount function relates to a preference for combining good with good. Let $[\tilde{\tau}_1; \tilde{\tau}_2]$ denote the risk that yields random variables $\tilde{\tau}_1$ and $\tilde{\tau}_2$ with equal probability. For some positive delay $d > 0$, consider the time risks $(\tilde{\tau}_A^{(2)}, x) := ([s+d; l+0], x)$ and $(\tilde{\tau}_B^{(2)}, x) := ([s+0; l+d], x)$; that is,

$$(\tilde{\tau}_A^{(2)}, x) \begin{array}{l} \overset{\frac{1}{2}}{\nearrow} (s+d, x) \\ \\ \underset{\frac{1}{2}}{\searrow} (l+0, x) \end{array} \qquad\qquad (\tilde{\tau}_B^{(2)}, x) \begin{array}{l} \overset{\frac{1}{2}}{\nearrow} (s+0, x) \\ \\ \underset{\frac{1}{2}}{\searrow} (l+d, x). \end{array}$$

Since less delay is better, as before, l is bad compared to the good s. Likewise, d is bad compared to the good 0, which is depicted in the down-state of lottery $(\tilde{\tau}_A^{(2)}, x)$ precisely to make this connection salient. In this sense, $(\tilde{\tau}_A^{(2)}, x) \succeq (\tilde{\tau}_B^{(2)}, x)$ is consistent with a preference for combining good with bad in each of its two states of nature, while $(\tilde{\tau}_B^{(2)}, x) \succeq (\tilde{\tau}_A^{(2)}, x)$ is consistent with a preference for combining good with good (in its up-state) as well as bad with bad (in its down-state). For the sake of brevity, in the following we will omit the "as well as bad with bad" part when unnecessary. By Observation 1, the DM prefers to combine good with good in the above situation if and only if

$$\frac{1}{2}\rho(s+d) + \frac{1}{2}\rho(l) \leq \frac{1}{2}\rho(s) + \frac{1}{2}\rho(l+d) \iff \rho(s+d) - \rho(s) \leq \rho(l+d) - \rho(l).$$

Dividing by d and letting d approach zero implies $\rho'(s) \leq \rho'(l)$ so that for l close to s, we obtain $\rho''(s) \geq 0$, that is, convex discounting. This result motivates the following definition of risk-averse (and risk-seeking) discounting:[3]

[3]It is easily observed that a preference for combining good with bad implies a concave rather than a convex discount function,

Definition 1. (Risk-averse discounting). *A DM exhibits risk-averse (risk-seeking) discounting if and only if* $(\tilde{\tau}_A^{(2)}, x) \succeq (\preceq)(\tilde{\tau}_B^{(2)}, x)$ *for all* s, l, d.

3.3 Prudent discounting

Let $\tilde{t}_1$ and $\tilde{t}_2$ denote independent zero-mean delay risks, and throughout let $s, l, \tilde{t}_1$ and $\tilde{t}_2$ be such that the support of total delay remains positive. Let $(\tilde{\tau}_A^{(3)}, x) := ([s + \tilde{t}_1; l + 0], x)$ and $(\tilde{\tau}_B^{(3)}, x) := ([s + 0; l + \tilde{t}_1], x)$, that is,

$$(\tilde{\tau}_A^{(3)}, x) \begin{array}{l} \overset{\frac{1}{2}}{\nearrow} \ (s + \tilde{t}_1, x) \\ \underset{\frac{1}{2}}{\searrow} \ (l + 0, x) \end{array} \qquad (\tilde{\tau}_B^{(3)}, x) \begin{array}{l} \overset{\frac{1}{2}}{\nearrow} \ (s + 0, x) \\ \underset{\frac{1}{2}}{\searrow} \ (l + \tilde{t}_1, x). \end{array}$$

$(\tilde{\tau}_A^{(3)}, x) \succeq (\tilde{\tau}_B^{(3)}, x)$ implies that the DM prefers an unavoidable zero-mean delay risk in the state in which delay is good. Consider a DM who prefers to combine good with good. As we have seen in the previous subsection, her discount function must be convex so that she is delay risk-seeking. Therefore, an additional zero-mean delay risk is good relative to zero additional delay. Accordingly, she prefers $(\tilde{\tau}_A^{(3)}, x)$, which combines the good short delay s with the good zero-mean delay risk in the up-state, over $(\tilde{\tau}_B^{(3)}, x)$, which has one good (and one bad) in each state. Let us compare this behavior with that of a DM who prefers to combine good with bad. His discount function is concave, which makes him time risk-averse, and thus a zero-mean delay risk is considered bad. A preference for combining good with bad, therefore, *also* implies a preference for $(\tilde{\tau}_A^{(3)}, x)$ over $(\tilde{\tau}_B^{(3)}, x)$. As in the utility paradigm (Crainich et al. 2013), we define prudent discounting as the time risk lottery preference that follows from both the preference for combining good with good and the preference for combining good with bad. Prudent discounting means that an outcome is discounted less upon increasing delay risk when current delay is low as compared to when it is high.

Definition 2. (Prudent discounting). *A DM exhibits prudent (imprudent) discounting if and only if* $(\tilde{\tau}_A^{(3)}, x) \succeq (\preceq)(\tilde{\tau}_B^{(3)}, x)$ *for all* $s, l, \tilde{t}_1$.

We now establish an equivalence between prudent discounting and the sign of the third derivative of the discount function. To this means, we define the *discount premium* in analogy to the utility premium of

which is incompatible with the discount function being positive (if defined on the whole positive real line), as noted by DeJarnette et al. (2018). While the subsequent analysis goes through irrespectively of this observation, preferences for combining good with bad may be regarded more important in the case of undesirable outcomes; see Section 5.

Friedman and Savage (1948) as the change in the (expected) discount factor at time t that results from taking the zero-mean time risk $\tilde{t}_1$,

$$D_{\tilde{t}_1}(t) := \mathbb{E}[\rho(t+\tilde{t}_1)] - \rho(t).$$

From our previous discussion, the discount premium is positive if and only if the discount function is convex. From Observation 1, the preference for $(\tilde{\tau}_A^{(3)}, x)$ over $(\tilde{\tau}_B^{(3)}, x)$ is equivalent to

$$\frac{1}{2}\mathbb{E}[\rho(s+\tilde{t}_1)] + \frac{1}{2}\rho(l) \geq \frac{1}{2}\rho(s) + \frac{1}{2}\mathbb{E}[\rho(l+\tilde{t}_1)] \Longleftrightarrow D_{\tilde{t}_1}(s) \geq D_{\tilde{t}_1}(l),$$

which is true for all $s < l$ if and only if the discount premium is decreasing: $D'_{\tilde{t}_1}(s) \leq 0$. We have

$$D'_{\tilde{t}_1}(s) \leq 0 \Longleftrightarrow \mathbb{E}[\rho'(s+\tilde{t}_1)] - \rho'(s) \leq 0,$$

which holds by Jensen's inequality if and only if ρ' is concave at s; that is, $\rho'''(s) \leq 0$. Therefore, discount prudence is characterized by a *negative* third derivative, which is opposite to the case of utility prudence, and ultimately a consequence of the fact that discount functions are decreasing while utility functions are increasing.

3.4 Temperate discounting

Fourth-order discounting preferences can be characterized through preferences over the lotteries $(\tilde{\tau}_A^{(4)}, x) := ([s+\tilde{t}_1+0; s+0+\tilde{t}_2], x)$ and $(\tilde{\tau}_B^{(4)}, x) := ([s+\tilde{t}_1+\tilde{t}_2; s+0+0], x)$; that is,

$$(\tilde{\tau}_A^{(4)}, x) \begin{cases} \frac{1}{2} & (s+\tilde{t}_1+0, x) \\ \frac{1}{2} & (s+0+\tilde{t}_2, x) \end{cases} \qquad (\tilde{\tau}_B^{(4)}, x) \begin{cases} \frac{1}{2} & (s+\tilde{t}_1+\tilde{t}_2, x) \\ \frac{1}{2} & (s+0+0, x). \end{cases}$$

If the DM prefers to combine good with good (good with bad), then by Section 3.2 the zero-mean risks are good (bad) relative to the zero-delays. Accordingly, the former prefers $(\tilde{\tau}_B^{(4)}, x)$, while the latter prefers $(\tilde{\tau}_A^{(4)}, x)$. We define temperate discounting as the trait that follows from a preference for combining good with bad.

Definition 3. (Temperate discounting). *A DM exhibits temperate (intemperate) discounting if and only if* $(\tilde{\tau}_A^{(4)}, x) \succeq (\preceq)(\tilde{\tau}_B^{(4)}, x)$ *for all* $s, \tilde{t}_1, \tilde{t}_2$.

Temperance (intemperance) means that time risks are *mutually aggravating (mutually enticing)* so that the DM prefers to disaggregate (aggregate) them across states of nature (Kimball 1993). Similarly to the previous calculations, intemperate discounting can be shown to be equivalent to $\rho'''' \geq 0$.[4]

3.5 Higher orders and anti-mixed discounting

We showed that a preference for combining good with good implies that DMs are risk-seeking ($\rho'' \geq 0$), prudent ($\rho''' \leq 0$), and intemperate ($\rho'''' \geq 0$) toward time risk. A preference for combining good with bad implies that DMs are risk-averse ($\rho'' \leq 0$), prudent ($\rho''' \leq 0$), and temperate ($\rho'''' \leq 0$). By adapting the lottery nesting procedure outlined in Eeckhoudt and Schlesinger (2006), these results can be extended to characterize the signs of even higher derivatives of the discount function.[5]

What are the signs of the derivatives of practically relevant (decreasing) discount functions? For discounting that is exponential (Samuelson 1937), Herrnstein's (1961) $\rho(t) = t^{-1}$, actually hyperbolic (Harvey 1986, 1995; Mazur 1987), generalized hyperbolic (Loewenstein and Prelec 1992), and for the continuous-time extension by Harris and Laibson (2013) of quasi-hyperbolic (Phelps and Pollak 1968; Laibson 1997), the signs of derivatives alternate in sign.[6] As Caballé and Pomansky (1996) call (the risk preferences implied by) an increasing utility function with derivatives alternating in sign *mixed (risk-averse)* let us call (the risk preferences implied by) a decreasing discount function with derivatives alternating in sign *anti-mixed (risk-averse).* In the title of their article, Brockett and Golden (1987) have noted that "[a]ll commonly

[4]By Observation 1, intemperate discounting is equivalent to

$$\frac{1}{2}\mathbb{E}[\rho(s+\tilde{t}_1)] + \frac{1}{2}\mathbb{E}[\rho(s+\tilde{t}_2)] \leq \frac{1}{2}\mathbb{E}[\rho(s+\tilde{t}_1+\tilde{t}_2)] + \frac{1}{2}\rho(s)$$
$$\iff \mathbb{E}[\rho(s+\tilde{t}_1)] - \rho(s) \leq \mathbb{E}[\rho(s+\tilde{t}_2+\tilde{t}_1)] - \mathbb{E}[\rho(s+\tilde{t}_2)] \iff D_{\tilde{t}_1}(s) \leq \mathbb{E}[D_{\tilde{t}_1}(s+\tilde{t}_2)].$$

By Jensen's inequality, the last inequality is equivalent to $D''_{\tilde{t}_1}(s) \geq 0$. We have

$$D''_{\tilde{t}_1}(s) \geq 0 \iff \mathbb{E}[\rho''(s+\tilde{t}_1)] - \rho''(s) \geq 0,$$

and invoking Jensen's inequality once more yields that ρ'' itself must be convex at s, that is, $\rho''''(s) \geq 0$.

[5]The details of this exercise are left for future research. Lajeri-Chaherli (2004) defines *edginess* as the fifth derivative of the utility function being positive. During his talk at the Conference in Honor of Louis Eeckhoudt in 2011, Miles Kimball suggested that the term "bentness" refers to a negative sixth derivative of the utility function. Harris Schlesinger, then, remarked that earlier that day "someone at breakfast" already had suggested the term "kimballesque."

[6]Bleichrodt et al. (2009) offer a comprehensive overview of parametric forms of discount functions. The two new discount functions proposed in that article also have derivatives of alternating sign if one assumes decreasing impatience. Ebert et al. (2016) offer further results and interpretations of discount functions whose derivatives alternate in sign.

used utility functions" are mixed risk-averse; above we have noted that commonly used discount functions are anti-mixed risk-averse. And while Eeckhoudt et al. (2009) have shown that the intuition behind mixed risk aversion is (i) more (outcome) is better and (ii) a preference for combining good with bad, we have shown that anti-mixed discounting of desirable outcomes is consistent with (i) less (delay) is better and (ii) a preference for combining good with good.

4 Applications

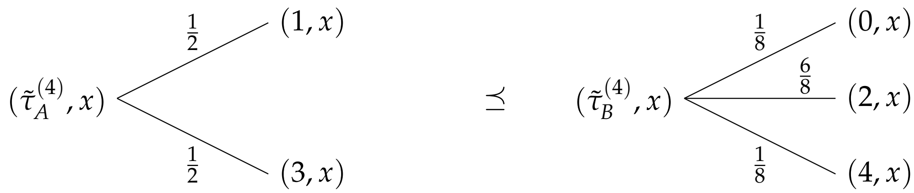
4.1 Preferences toward skewed and leptokurtic time risks

This section illustrates that prudent and temperate discounting can determine preferences toward skewed and leptokurtic delay risks; whether the DM is delay risk-seeking or delay risk-averse plays no role in this regard. First, consider the prudence lottery pair from Section 3.3 with $s = 1$, $l = 2$, and zero-mean risk $\tilde{t} = [1; -1]$; that is, $(\tilde{\tau}_A^{(3)}, x) = ([1 + [1; -1]; 2], x)$, and $(\tilde{\tau}_B^{(3)}, x) = ([1; 2 + [1; -1]], x)$. After reducing these compound lotteries, it follows that discount prudence ($\rho''' \leq 0$) implies the preference

$$(\tilde{\tau}_A^{(3)}, x) \; \begin{array}{l} \xrightarrow{\frac{1}{4}} (0, x) \\ \xrightarrow[\frac{3}{4}]{} (2, x) \end{array} \qquad \succeq \qquad (\tilde{\tau}_B^{(3)}, x) \; \begin{array}{l} \xrightarrow{\frac{1}{4}} (3, x) \\ \xrightarrow[\frac{3}{4}]{} (1, x). \end{array}$$

It is easy to verify that the delays of both choices have equal mean and variance. $(\tilde{\tau}_A^{(3)}, x)$, however, is left-skewed, while $(\tilde{\tau}_B^{(3)}, x)$, is right-skewed. The example above was chosen so as to recover a lottery pair of the type proposed in Mao (1970) and Ebert and Wiesen (2011), which—in the outcome risk case—these authors used to test skewness preference. While outcome prudence relates to a preference for positive skewness, discount prudence relates to a preference for negative skewness—another consequence of the fact that more outcome is good, while more delay is bad. Noting further that the mean of both delays is 1.5, the negatively skewed $(\tilde{\tau}_A^{(3)}, x)$ is lottery-like in the sense that it features a much shorter delay with low probability and a mildly larger delay with high probability. Therefore, both discount and outcome prudence are related to a preference for lottery-like risks.

Following Menezes and Wang (2005), one can show that the temperance lotteries are *outer risk increases/contractions* of one another, meaning that they have equal mean, variance, and skewness but differ in their kurtosis. Just as a convex discount function implies risk-seeking, intemperate discounting implies outer risk-seeking. As an example, an intemperate DM prefers $(\tilde{\tau}_B^{(4)}, x) = ([2 + [1; -1] + [1; -1]; 2]], x)$ over the less leptokurtic $(\tilde{\tau}_A^{(4)}, x) = ([2 + [1; -1]; 2 + [1; -1]], x)$:



4.2 Investment under delay risk

Let $\mathcal{X} = [0, \infty)$. Assume the support of $\tilde{\tau}$ to be contained in $\{1, \ldots, T\}$ and consider an investor with differentiable, strictly increasing, and concave per-period utility u with $u(0) = 0$, who derives utility from consumption in periods $\{0, 1, \ldots, T\}$. Consider an investment opportunity that is characterized by $(\tilde{\tau}, x) \in \Delta$, paying off $x > 0$ for each unit invested when exiting the company at the yet uncertain time $\tilde{\tau}$. Without investing, his consumption in each period is $c > 0$. By giving up $\alpha \in [0, c]$ of current ($t = 0$) consumption, the DM can increase her consumption to αx in the yet unknown exit period $\tilde{\tau}$. Accordingly, his objective is to maximize

$$V(\alpha) = u(c - \alpha) + \sum_{\tau=1}^{T} \mathbb{P}[\tilde{\tau} = \tau] \left(\sum_{t=1, t \neq \tau}^{T} \rho(t) u(c) + \rho(\tau) u(c + \alpha x) \right),$$

which can be rewritten as

$$\begin{aligned} V(\alpha) &= u(c - \alpha) + \sum_{\tau=1}^{T} \mathbb{P}[\tilde{\tau} = \tau] \left(\left(\sum_{t=1}^{T} \rho(t) - \rho(\tau) \right) u(c) + \rho(\tau) u(c + \alpha x) \right) \\ &= u(c - \alpha) + \sum_{\tau=1}^{T} \mathbb{P}[\tilde{\tau} = \tau] \left(\sum_{t=1}^{T} \rho(t) \right) u(c) + \sum_{\tau=1}^{T} \mathbb{P}[\tilde{\tau} = \tau] \left(-\rho(\tau) u(c) + \rho(\tau) u(c + \alpha x) \right) \\ &= u(c - \alpha) + u(c) \sum_{t=1}^{T} \rho(t) + \mathbb{E}[\rho(\tilde{\tau})] \left(u(c + \alpha x) - u(c) \right). \end{aligned} \tag{5}$$

The first-order condition is given by

$$V'(\alpha) = -u'(c - \alpha) + \mathbb{E}[\rho(\tilde{\tau})] x u'(c + \alpha x) = 0.$$

Let us compare the investment decision under different payout time distributions $\tilde{\tau}_i$ and denote the corresponding objective functions by V_i and the optimal investment level by α_i^* ($i = A, B$). For arbitrary α, since x and marginal utility are positive,

$$\mathbb{E}[\rho(\tilde{\tau}_A)] \leq \mathbb{E}[\rho(\tilde{\tau}_B)] \Longleftrightarrow V_A'(\alpha) \leq V_B'(\alpha) \Longrightarrow \alpha_A^* \leq \alpha_B^*,$$

where the last conclusion follows from the fact that the V_i' are decreasing so that $V_A'(\alpha)$ intersects with $\alpha = 0$ at a smaller value of α. Therefore, the first-order condition of the maximization problem is fulfilled at a smaller value of α. If ρ is anti-mixed, the DM will prefer the project whose exit time is earlier than (a mean-preserving spread of, a downside risk decrease of, an outer risk increase of) that of the comparison project because his discount function is decreasing (risk-seeking, prudent, intemperate).

4.3 Comparative time risk-seeking and the Arrow-Pratt measure of a discount function

For $t \geq 0$ consider a time risk $(\tilde{\tau}, x) \in \Delta$ such that $t + \tilde{\tau} \geq 0$ and $E[\tilde{\tau} = 0]$. The *delay risk premium* π_d of $(\tilde{\tau}, x)$ is defined as the solution to

$$U(t - \pi_d, x) = U(t + \tilde{\tau}, x) \Longleftrightarrow \rho(t - \pi_d) = \mathbb{E}[\rho(t + \tilde{\tau})], \tag{6}$$

where the equivalence is a consequence of Observation 1. We observe that the risk premium $\pi_d \equiv \pi_d(t)$ is a function of time only (independent of the size of the outcome). Assuming convex discounting, the interpretation of the delay risk premium is as follows. Receiving the desirable outcome x at time $t - \pi_d(t)$ for sure is as good as waiting the additional time $\pi_d(t)$ to receive the outcome sooner or later, that is, with the additional good delay risk $\tilde{\tau}$. Following the analysis of de Finetti (1952), Pratt (1964), and Arrow (1965), for small risks $\tilde{\tau}$ with variance σ^2, we obtain the approximation

$$\pi_d(t) \approx \frac{1}{2}\sigma^2 A_\rho(t) \tag{7}$$

where $A_\rho(t) = -\frac{\rho''(t)}{\rho'(t)}$ denotes the Arrow-Pratt measure of the discount function ρ, that is, the coefficient of absolute delay-risk-seeking. For anti-mixed discount functions $A_\rho(t) > 0$ so that also $\pi_d(t) > 0$. The time t Arrow-Pratt measure of a discount function thus describes the DM's *willingness to wait (WTW)* until time t for a desirable outcome in order to receive it at an uncertain rather than at a certain time. We state the following result without proof, because with Observation 1 in mind it is a straightforward adaptation from results in Pratt (1964); see also Gollier (2001) [p. 20–21].

Proposition 1. (Comparative time risk-seeking toward desirable outcomes). *Consider DMs $i = 1, 2$ with discount functions ρ_i, corresponding Arrow-Pratt measures $A_i(t)$, and delay risk premia $\pi_{d,i}$ for the time risk $\tilde{\tau}$. DM 2 is* more time risk-seeking *than DM 1 in the sense of the following three equivalent statements:*

$$A_2(t) \geq A_1(t), \forall t, x. \tag{I}$$

$$\textit{DM 2 accepts all time risks } \tilde{t} \textit{ at } t \textit{ that DM 1 accepts at } t, \forall t, x. \tag{II}$$

$$\textit{DM 2 is willing to wait longer for time risk } \tilde{\tau} \textit{ at } t \textit{ than DM 1: } \pi_{d,2} \geq \pi_{d,1}, \forall t, x. \tag{III}$$

We close this section by analyzing the connection between discount prudence and *decreasing absolute time risk-seeking (DATRS)*; that is,

$$\frac{\partial}{\partial t} A(t) \leq 0 \Longleftrightarrow \rho'''(t)\rho'(t) \geq (\rho''(t))^2. \tag{8}$$

Since $\rho' < 0$, it follows that prudent discounting ($\rho''' \leq 0$) is necessary for DATRS. This is similar to atemporal EU, where, since $u' > 0$, outcome prudence ($u''' \geq 0$) is necessary for decreasing absolute risk aversion (DARA).

4.4 Precautionary patience: Prudent behavior toward delay risk

Consider a DM who receives two payments of size x (each granting utility $u(x) > 0$), one at time $s \equiv 0$ and one at time $l > 0$. Consider the opportunity to delay the first payment by $\hat{d}$ days in order to receive the second payment $\hat{d}$ days sooner. The DM's problem is given by

$$\max_{\hat{d} \in [0,l]} \rho(\hat{d})u(x) + \rho(l - \hat{d})u(x).$$

Assuming that ρ is convex, corner solutions prevail. The DM is thus indifferent between $\hat{d} = 0$ and $\hat{d} = l$, each of which grants EDU $u(x)(\rho(0) + \rho(l))$.[7] Intuitively, since the DM is time risk-seeking, she wants

[7] We assume that for $\hat{d} = \frac{l}{2}$, the DM frames the payments narrowly (i.e., receives utility $u(x) + u(x)$). Alternatively, if $\mathcal{X} = [0, \infty)$, we could assume that u is concave so that $u(2x) \leq u(x) + u(x)$. Further, for simplicity we ignore the issue of negative time, which is easily resolved by replacing current time $s = 0$ with some $s > 0$ and restricting the support of $\tilde{t}$ to $[-s, \infty)$.

both payments to be as far apart from each other as possible (and she doesn't care whether she receives the "first" payment of x now and the "second" payment of x later at l, or vice versa).

Next, suppose that the time of when the future payment is made becomes uncertain, so that l is replaced by $l+\tilde{t}$, where $\tilde{t}$ is some zero-mean risk. The DM's problem becomes

$$\max_{\hat{d}\in[0,l]} u(x)\left(\rho(\hat{d})+\mathbb{E}[\rho(l-\hat{d}+\tilde{t})]\right).$$

The convexity of ρ again excludes interior solutions. For $\hat{d}=0$, the DM's utility is given by

$$u(x)\left(\rho(0)+\mathbb{E}[\rho(l+\tilde{t})]\right),$$

while for $\hat{d}=l$ it is given by

$$u(x)\left(\rho(l)+\mathbb{E}[\rho(\tilde{t})]\right).$$

The DM strictly prefers $\hat{d}=l$ over $\hat{d}=0$ if

$$\frac{1}{2}\rho(l)u(x)+\frac{1}{2}\mathbb{E}[\rho(\tilde{t})]u(x)>\frac{1}{2}\rho(0)u(x)+\frac{1}{2}\mathbb{E}[\rho(l+\tilde{t})]u(x),$$

which describes a (strict) prudence lottery preference $(\tilde{\tau}_A^{(3)},x)$ over $(\tilde{\tau}_B^{(3)},x)$; see Section 3.3. Therefore, the DM is willing to delay a delay-certain payment in order to receive a time-uncertain payment sooner if and only if she is a strictly prudent discounter ($\rho'''<0$). Recall that strict utility prudence ($u'''>0$) implies precautionary saving in EDU, that is, the willingness to increase saving toward uncertain states of nature (e.g., Crainich et al. 2013). Therefore, both discount- and utility-prudent behavior is precautionary in the sense that it results in a better constitution of the DM (higher wealth and less delay, respectively) in the state that is risky.

5 The case of undesirable outcomes

5.1 The dual discount function

Given a discount function ρ, let us define the *dual discount function* by $\hat{\rho}(t) := 1 - \rho(t)$. While the discount function describes what remains of utility as time goes by (i.e., the *discount factor*), the dual discount function describes the corresponding *discount.* For negative utility, a larger discount is better. In line with this idea, the case $u(x) < 0$ of equivalence (ii) in Observation 1 can be rewritten as

$$(\tilde{\tau}_1, x) \succeq (\tilde{\tau}_2, x) \Longleftrightarrow \mathbb{E}[\hat{\rho}(\tilde{\tau}_1)] \geq \mathbb{E}[\hat{\rho}(\tilde{\tau}_2)]. \tag{9}$$

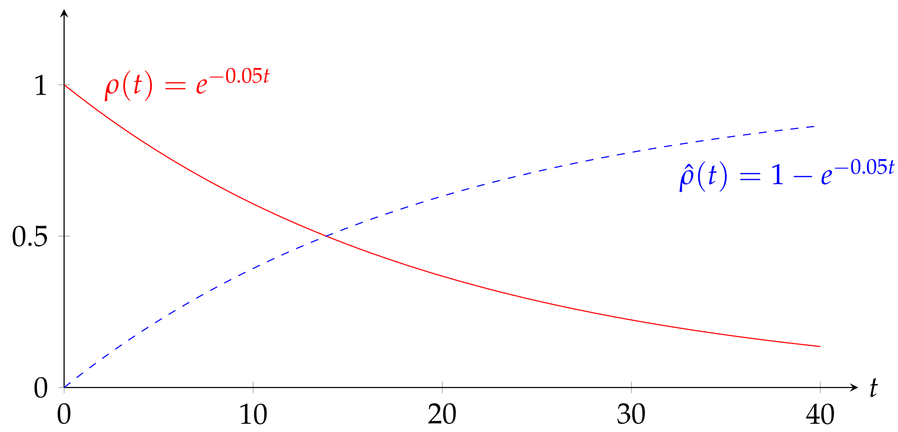
For anti-mixed discount functions ρ, the dual discount function $\hat{\rho}$ is mixed. Therefore, because the shape of $\hat{\rho}$ is the same as that of a "typical" u and because the right-hand inequality in equation (9) is in the same direction as that in equivalence (i) in Observation 1, delay risk preferences toward undesirable outcomes are even more similar to the well-studied outcome risk preferences than they were in the case of desirable outcomes. The anti-mixed discounting of undesirable outcomes that are only a matter of time is risk-averse, prudent, and temperate, and consistent with (i) more delay is better and (ii) a preference for combining good with bad; see Eeckhoudt et al. (2009) for details.

5.2 Exponential discounting of undesirable outcomes that are only a matter of time

This section focuses on the important special case of exponential discounting and offers a new characterization thereof in terms of time risk preferences. For $\rho(t) = e^{-\delta t}$ with $\delta > 0$, the dual discount function is given by $\hat{\rho}(t) = 1 - e^{-\delta t}$; see Figure 1 for an illustration. We observe that $\hat{\rho}$ is structurally identical to the well-known exponential (constant absolute risk-averse, CARA) utility function. From Observation 1 it follows that everything we know about CARA risk preferences applies *exactly* for time risk preferences toward undesirable outcomes when discounting is exponential. In particular, we obtain a new characterization of exponential discounting: An individual is an exponential discounter if and only if her time risk preferences toward the arrival of an undesirable outcome exhibit constant absolute time risk aversion (CATRA).

How is CATRA to be interpreted? The Arrow-Pratt measure at time t (i.e., the measure of absolute time risk aversion at time t; recall Section 4.3) now describes the DM's *willingness to accelerate* the occurrence of the undesirable outcome in order to receive it at a certain rather than at an uncertain date. CATRA thus

Figure 1: The exponential (dual) discount function



Notes. Figure 1 plots an exponential discount function $\rho(t)$ with a discount rate of $\delta = 0.05$ and the corresponding dual discount function $\hat{\rho} = 1 - \rho(t)$. The dual discount function is mathematically identical to CARA utility u over consumption x with the coefficient of absolute risk aversion being equal to the discount rate: $u(x) = 1 - e^{-0.05x}$. For that reason, risk preferences toward the delay of an undesirable outcome are like outcome risk preferences with constant absolute risk aversion.

means that the willingness to accelerate is calendar-time independent. As an example, consider a DM who has a "skeleton in the closet," knowing that it will come out sooner or later. If and only if the DM discounts exponentially, his willingness to accelerate the coming out in exchange for removing the uncertainty thereof is independent of whether he expects the coming out to be in, say, one or two years' time.

5.3 Willingness to pay (WTP) and net present value (NPV) under time risk

The net present value (NPV) rule is the most widely used tool in investment analysis. The NPV rule says that an investment to prevent a future loss should be made if and only if the cost of the investment is lower than its NPV. The NPV of avoiding a loss coincides with the willingness to pay (WTP) to avoid the loss if the social planner is outcome risk-neutral. In this section, we study a social planner's WTP to avoid an unavoidable loss that is only a matter time. We study how this WTP changes under different time risks and discuss a generalized version of the seminal NPV formula that incorporates time risk.

Let $\mathcal{X} = [0, \infty)$. Consider a social planner who experiences strictly increasing per-period utility u of baseline consumption $c > 0$ today (at $t = 0$) as well as in each future period $t = 1, 2, \ldots, T$, where $u(0) = 0$. She faces a non-degenerate time risk $(\tilde{\tau}_2, L) \in \Delta$, with the support of $\tilde{\tau}_2$ contained in $\{1, 2, \ldots, T\}$ and $0 < L < c$, so that consumption in the yet unknown period $\tilde{\tau}_2$ is reduced to $c - L$. Suppose, first, that the social planner cannot prevent the loss, but she can invest today in a technology that will transform $\tilde{\tau}_2$ into $\tilde{\tau}_1$. Assume that the support of $\tilde{\tau}_1$ is also contained in $\{1,2,\ldots,T\}$ and that all time risks considered have an integer expectation. (Then, $\tilde{\tau}_1 \equiv \mathbb{E}[\tilde{\tau}_2]$ is the special case in which time risk is removed entirely.) Let

$$v(\tau) := \sum_{t=1, t\neq\tau}^{T} \rho(t)u(c) + \rho(\tau)u(c-L), \tag{10}$$

denote the (not necessarily exponentially) discounted utility of future consumption (i.e., of consumption from period $t = 1$ onward), given that the loss is realized at time τ. The planner's WTP for the technology that transfers $\tilde{\tau}_2$ into $\tilde{\tau}_1$, denoted by π, is defined implicitly via

$$u(c-\pi) + \mathbb{E}[v(\tilde{\tau}_1)] = u(c) + \mathbb{E}[v(\tilde{\tau}_2)]. \tag{11}$$

Rearrangements similar to those detailed in the derivation of equation (5) yield

$$\mathbb{E}[\rho(\tilde{\tau}_2)] - \mathbb{E}[\rho(\tilde{\tau}_1)] = \frac{u(c) - u(c-\pi)}{u(c) - u(c-L)}. \tag{12}$$

We denote the right-hand side of this equality by $\Delta u(\pi) = \frac{u(c)-u(c-\pi)}{u(c)-u(c-L)}$, which is a strictly increasing one-to-one mapping of π. Moreover, $\Delta u(\pi) > 0 \iff \pi > 0$, and therefore

$$\mathbb{E}[\rho(\tilde{\tau}_2)] > \mathbb{E}[\rho(\tilde{\tau}_1)] \iff \pi > 0. \tag{13}$$

The inequality on the left-hand side is familiar by now, and equivalence (13) shows that it also determines the WTP to reduce the time risk of a loss. As an example, when $\tilde{\tau}_2$ is more left-skewed than $\tilde{\tau}_1$ (as when $\tilde{\tau}_2 = \tilde{\tau}_B^{(3)} + 1$ and $\tilde{\tau}_1 = \tilde{\tau}_A^{(3)} + 1$, as in Section 4.1), then discount prudence ensures a positive WTP, $\pi > 0$. As another example, convex discounting is equivalent to a positive WTP for facing the loss at a certain time over facing it at an uncertain time with equal expectation, which follows from equation (13), when $\tilde{\tau}_1 \equiv \mathbb{E}[\tilde{\tau}_2]$. Because we will refer to this case again later, let us denote the premium of eliminating a time risk $\tilde{\tau}$ entirely (i.e., the case $\tilde{\tau}_2 = \tilde{\tau}$ and $\tilde{\tau}_1 = \mathbb{E}[\tilde{\tau}]$) by $\pi_{\tilde{\tau}}$, in which case equation (12) becomes

$$\mathbb{E}[\rho(\tilde{\tau})] - \rho(\mathbb{E}[\tilde{\tau}]) = \Delta u(\pi_{\tilde{\tau}}). \tag{T}$$

Next, let us study the WTP to avoid not only the time risk $(\tilde{\tau}, L)$ but the loss entirely. We denote the WTP for it by $\pi_{\tilde{\tau}L}$, because a technology that eliminates L also makes the realization of $\tilde{\tau}$ irrelevant. It is defined via

$$u(c - \pi_{\tilde{\tau}L}) + \sum_{t=1}^{T} \rho(t) u(c) = u(c) + \mathbb{E}[v(\tilde{\tau})],$$

which can be transformed into

$$\mathbb{E}[\rho(\tilde{\tau})] = \Delta u(\pi_{\tilde{\tau}L}). \tag{TL}$$

Before we interpret equation (TL), note that avoiding the loss entirely in case of a degenerate time risk, $\tilde{\tau} \equiv \mathbb{E}[\tilde{\tau}]$, yields a WTP (which we denote by π_L), given by

$$\rho(\mathbb{E}[\tilde{\tau}]) = \Delta u(\pi_L). \tag{L}$$

By combining equations (TL), (L), (T), one can observe that

$$\Delta u(\pi_{\tilde{\tau}L}) = \Delta u(\pi_L) + \Delta u(\pi_{\tilde{\tau}}). \tag{14}$$

That is, the WTP to prevent a loss at an uncertain time can be additively decomposed into a component attributable to the loss itself and a component attributable to the associated time risk. This result holds for all increasing utility functions, but it is most crisply illustrated for the case of a risk-neutral social planner. This case is also economically important, since the WTP coincides with the net present value (NPV) of avoiding the loss. In particular, note that under risk-neutrality, we have $\Delta u(\pi) = \frac{\pi}{L}$. Then,

$$\pi_{\tilde{\tau}L} = \pi_L + \pi_{\tilde{\tau}}, \tag{NPV}$$

and the respective contributions are given by

$$\pi_{\tilde{\tau}L} = L \cdot \mathbb{E}[\rho(\tilde{\tau})], \; \pi_L = L \cdot \rho(\mathbb{E}[\tilde{\tau}]), \text{ and } \pi_{\tilde{\tau}} = L \cdot (\mathbb{E}[\rho(\tilde{\tau})] - \rho(\mathbb{E}[\tilde{\tau}])).$$

π_L is the NPV, as usually considered in the literature, that is, the NPV of not incurring the loss of L in the (certain) period $\mathbb{E}[\tilde{\tau}]$. It is simply given by the loss itself, discounted by the time at which it will occur. In the presence of time risk and when using any of the commonly used (anti-mixed) discount functions, this NPV needs to be adjusted upward. The upward adjustment $\pi_{\tilde{\tau}}$ is given by the product of the loss size and the discount premium $D_{\tilde{\tau}-\mathbb{E}[\tilde{\tau}]}(\mathbb{E}[\tilde{\tau}]) = \mathbb{E}[\rho(\tilde{\tau})] - \rho(\mathbb{E}[\tilde{\tau}])$, as defined in Section 3.3. This discount premium describes the change in the (expected) discount factor that results from making the time of the loss's occurrence risky; that is, from replacing $\mathbb{E}[\tilde{\tau}]$ by $\tilde{\tau}$. Ignoring $\pi_{\tilde{\tau}}$ or the discount premium results in systematic underinvestment in measures against losses that are only a matter of time.

As an illustration of this section's results, consider the case in which the loss L is the consequence of an environmental catastrophe due to climate change (e.g., a flood due to a rise in sea level). A large body of literature is concerned with determining the discount rate that is appropriate for discounting the costs and benefits of climate change abatement activities; see, for example, Gollier (2014) and references therein. Our results clarify—for given discount functions with arbitrary discount rates—how time risk can be incorporated in cost-benefit analysis, simply by adding $\pi_{\tilde{\tau}}$ to the NPV. Taking into account time risk when computing the NPV of an abatement investment can prevent the social planner from misallocating her resources today.

6 Discussion

In order to illustrate the duality between outcome and time risk preferences, we have assumed that the outcome that is only a matter of time is certain. Due to the EDU assumption that discounting and utility are multiplicatively separable, all results in this article generalize to *random outcomes* that are only a matter of time, that is, to elements $(\tilde{\tau}, \tilde{x}) \in \Delta$—as long as $\tilde{\tau}$ and $\tilde{x}$ are independent. In the investment application of Section 4.2, for example, we may assume that the payoff at the random exit time is also random—as long as this randomness is independent from the exit time. The case of general $(\tilde{\tau}, \tilde{x}) \in \Delta$, where a random payoff occurs at a random time and where these uncertainties may be dependent, is more complicated. The novel properties of the discount function identified in this article might remain important though—just as the properties of atemporal utility functions remain important once taking into account the widely neglected aspect of time risk.

Another generalization of this article's results concerns the assumption of EDU preferences itself. It is relatively straightforward to extend the time-outcome risk duality to discounted versions of rank-dependent utility (RDU, Quiggin 1982) and cumulative prospect theory (CPT, Tversky and Kahneman 1992).[8] DeJarnette et al. (2018) propose and axiomatize a generalization of EDU. Among other things, their generalized expected discounted utility (GEDU) model allows for more flexible attitudes toward time risk and intertemporal substitution. Another generalization of preferences concerns the relaxation of the multiplicative separability of time and outcome as assumed in EDU. For fixed $x \in \mathcal{X}$ and $(\tilde{\tau}, x) \in \Delta$, the preference functional $U(\tilde{\tau}, x)$ in equation (4) is univariate. Then, all results go through by identifying the appropriate conditions on U rather than on ρ. The study of (higher-order) preferences toward ambiguous receipt times (rather than risky receipt times), as considered in an experiment by Eliaz and Ortoleva (2015), constitutes yet another topic for future research.

Some of this article's results are amenable for empirical testing. Ebert (2018) investigates the prevalence of prudence and temperance toward certain, desirable outcomes. The described asymmetries between the (higher-order) time risk preferences toward desirable and undesirable outcomes (e.g., preferences for combining good with good versus preferences for combining good with bad, respectively) deserve investigation, just as does the analysis of preferences toward (independent) random outcomes that are only a matter of time. Measurements of the various (delay) risk premia defined in this paper could shed light on the relevance and intensity of time risk preferences; higher-order time risk premia (e.g., the time risk analogue to the prudence premium considered in Crainich and Eeckhoudt 2008) could be defined and measured as well. One might also put various behavioral implications of time risk preferences (e.g., for investment) to the test. Confirmations and violations of predictions obtained under EDU could guide the development of more general models, as discussed above.

[8] To see this for RDU, let us focus on desirable outcomes and consider the rank-dependent generalization of equation (4), where

$$U(\tilde{\tau}, x) = \int_{\mathbb{R}_+} w\left(\mathbb{P}(\rho(\tilde{\tau})u(x) > y)\right) \mathrm{d}y$$

for some strictly increasing weighting function $w : [0,1] \longrightarrow [0,1]$, with $w(0) = 0$ and $w(1) = 1$. Substituting $z := \frac{y}{u(x)}$, it follows that

$$U(\tilde{\tau}, x) = u(x) \int_{\mathbb{R}_+} w\left(\mathbb{P}(\rho(\tilde{\tau}) > z)\right) \mathrm{d}z =: u(x) RDU[\rho(\tilde{\tau})],$$

and the time-outcome risk duality follows just as in the EDU case. The derivation for CPT is similar but involves more notation. Upon defining a certain future date as a reference point (i.e., reference date), earlier arrivals of desirable outcomes could be felt as a time gain, while later arrivals could be felt as a time loss. A kink of the discount function at the reference date (with a steeper decline thereof to the right) would imply time loss aversion.

7 Conclusion

This article analyzes time risk preferences and their behavioral implications within EDU. A simple observation referred to as the *time-outcome risk duality* clarifies how known results about utility functions and outcome risk preferences can be given a correspondence in the realm of discount functions and time risk preferences. We defined prudent ($\rho''' \leq 0$) and intemperate ($\rho'''' \geq 0$) discounting and studied some first behavioral implications of these and other properties in situations where a certain outcome is only a matter of time. Further applications may be down the road, and other results on utility functions, as collected, for example, in Gollier (2001), may turn out to have important correspondences in the realm of discount functions.

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